KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

Politechnika Wrocławska

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

ROZWÓJ POTENCJAŁU I OFERTY DYDAKTYCZNEJ POLITECHNIKI WROCŁAWSKIEJ

Wrocław University of Technology

Advanced Informatics and Control

K. J. Burnham, V. E. Ersanilli

MATHEMATICS AND COMPUTING FOR CONTROL

Wrocław 2011

Projekt współfinansowany ze środków Unii Europejskiej w ramach
Europejskiego Funduszu Społecznego

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Reviewer: Michał Woźniak

ISBN 978-83-62098-38-5

Published by PRINTPAP Łódź, www.printpap.pl

Preface

This book is one of a series of Masters level texts which have been produced for taught modules within a common course designed for Advanced Informatics and Control. The new common course development forms a collaboration between Coventry University, United Kingdom and Wroclaw University of Technology, Poland. The new course recognises the complexity of new and emerging advanced technologies in informatics and control, and each text is matched to the topics covered in an individual taught module. The source of much of the material contained in each text is derived from lecture notes, which have evolved over the years, combined with illustrative examples which may well have been used by many other authors of similar texts that can be found. Whilst the sources of the material may be many, any errors that may be found are the sole responsibility of the authors.

In a book of this nature, scope and purpose it is possible only to select certain limited key issues and address these in the context of the overall landscape of advanced informatics and control engineering. The student is reminded at each stage of the need to be aware of the broader picture and that indeed a thorough and rigorous treatment would consume a lifetime to grasp all but a subset of that which might be encountered in practice. The text is designed around specific lectures, with each having a target aim and intended learning outcome. There are worked examples and exercises throughout as well as open ended exercises to allow students an opportunity to study further at their own pace. Wherever it is pertinent, readers are encouraged to check all results using the built in functions of Matlab / Simulink as they re-work the examples and pursue the exercises.

Contents (including subsections)

K J Burnham and V E Ersanilli

Control Theory and Applications Centre

Coventry University

Chapter 1: Introduction to Linear Algebra for Control

1.1. Matrix Algebra

Consider a system of n equations in m variables

$$y_1 = a_{11}x_1 + a_{12}x_2 \cdots a_{1m}x_m$$

$$y_2 = a_{21}x_1 + a_{22}x_2 \cdots a_{2m}x_m$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_n = a_{n1}x_1 + a_{n2}x_2 \cdots a_{nm}x_m$$

This can be expressed using matrix notation as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

The n rows by m column array is termed a matrix and is denoted $\boldsymbol{A}$. The two columns ($n \times 1$) and ($m \times 1$) are termed vectors and denoted $\boldsymbol{y}$ and $\boldsymbol{x}$, respectively. Hence the set of equations may be written in the compact form

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$$

The vector $\boldsymbol{y}$ is the result of multiplying the matrix $\boldsymbol{A}$ by the vector $\boldsymbol{x}$. Note that multiplication of a vector and a matrix is a special case of multiplying a matrix by a matrix (note that $\boldsymbol{x}$ is a matrix having m rows and 1 column). A vector $\boldsymbol{x}$ is also a column matrix $\boldsymbol{X}$. It is also possible to consider a scalar, a, as a 1 row by 1 column matrix. $\boldsymbol{A} = [a_{ij}]$ where a_{ij} is a typical element of the matrix $\boldsymbol{A}$.

1.1.1. Matrix operations

a) Addition and subtraction

Consider

$$y_1 = a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_m = a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n$$

and

$$z_1 = b_{11}x_1 + b_{12}x_2 \cdots + b_{1n}x_n$$

$$z_2 = b_{21}x_1 + b_{22}x_2 \cdots + b_{2n}x_n$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$z_m = b_{m1}x_1 + b_{m2}x_2 \cdots + b_{mn}x_n$$

Adding gives

$$y_1 + z_1 = (a_{11} + b_{11})x_1 \cdots + (a_{1n} + b_{1n})x_n$$

$$y_2 + z_2 = (a_{21} + b_{21})x_1 \cdots + (a_{2n} + b_{2n})x_n$$

$$\vdots \quad \vdots \quad \vdots$$

$$y_m + z_m = (a_{m1} + b_{m1})x_1 \cdots + (a_{mn} + b_{mn})x_n$$

leading to

$$\begin{bmatrix} y_1 + z_1 \\ \vdots \\ y_m + z_m \end{bmatrix} = \begin{bmatrix} (a_{11} + b_{11}) & \cdots & (a_{1n} + b_{1n}) \\ \vdots & & \vdots \\ (a_{m1} + b_{m1}) & \cdots & (a_{mn} + b_{mn}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{y} + \boldsymbol{z} = (\boldsymbol{A} + \boldsymbol{B})\boldsymbol{x} = \boldsymbol{C}\,\boldsymbol{x}$$

where

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$$

It is clear that two matrices can be added only when both are of the same dimensions: here m rows and n columns.

Furthermore

$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$

b) Multiplication

This may be defined in terms of substitution of one linear system of equations into another

e.g.

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}, \qquad \boldsymbol{x} = \boldsymbol{B}\,\boldsymbol{w}$$

so that

$$\boldsymbol{y} = \boldsymbol{A}(\boldsymbol{B}\,\boldsymbol{w})$$

$$= \boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{w}$$

$$= \boldsymbol{C}\,\boldsymbol{w}$$

where $\boldsymbol{C}$ is the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ defined as

$$\boldsymbol{C} = \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{1n}b_{n1}) & \cdots & (a_{11}b_{1p} + \cdots + a_{1n}b_{np}) \\ \vdots & & \vdots \\ (a_{m1}b_{11} + \cdots + a_{mn}b_{n1}) & \cdots & (a_{m1}b_{1p} + \cdots + a_{mn}b_{np}) \end{bmatrix}$$

where $\boldsymbol{A}$ has m rows and n columns, and $\boldsymbol{B}$ has n rows and p columns.

Note that for $\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B}$ to exist, the number of rows of $\boldsymbol{B}$ must be equal to the number of columns of $\boldsymbol{A}$.

In general, for the typical element c_{ij} in $\boldsymbol{C}$

$$c_{ij} = \underbrace{[a_{i1}\; a_{i2} \cdots a_{in}]}_{i^{\text{th}}\text{row}} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} j^{\text{th}}\text{column}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

If the product $\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B}$ exists the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable. Note that whilst $\boldsymbol{A}\,\boldsymbol{B}$ may be defined the product $\boldsymbol{B}\,\boldsymbol{A}$ is, in general, undefined. Even when both $\boldsymbol{A}\,\boldsymbol{B}$ and $\boldsymbol{B}\,\boldsymbol{A}$ exist it is not necessary that $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A}$.

Consider the case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both of the form

$$\boldsymbol{A} = [a_1 \cdots a_n], \quad \boldsymbol{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$ i.e. $\boldsymbol{A}$ is a row matrix and $\boldsymbol{B}$ is a column matrix.

Then

$$\boldsymbol{A}\,\boldsymbol{B} = [a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n,$$ is a 1×1 scalar

and $$\boldsymbol{B}\,\boldsymbol{A} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} [a_1 \cdots a_n] = \begin{bmatrix} b_1a_1 & \cdots & b_1a_n \\ \vdots & & \vdots \\ b_na_1 & \cdots & b_na_n \end{bmatrix}$$ is an $n \times n$ matrix.

Clearly $\boldsymbol{A}\,\boldsymbol{B} \neq \boldsymbol{B}\,\boldsymbol{A}$.

However, even when $\boldsymbol{A}$ and $\boldsymbol{B}$ are both square matrices of dimension *n* x *n*, hence both $\boldsymbol{A}\,\boldsymbol{B}$ and $\boldsymbol{B}\,\boldsymbol{A}$ exist, it is not always true that $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A}$. When $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A}$ the matrices are said to commute. Note that a 1×1 matrix (or scalar) is conformable and commutes with any matrix. With scalars the operation of division by a quantity is defined as multiplication by the reciprocal of that quantity, i.e. $c = \frac{a}{b} = ab^{-1} = b^{-1}a$ where $b^{-1} = \frac{1}{b}$. Extending this concept to matrices, it is first necessary to define the reciprocal, which is termed the inverse of the matrix.

1.1.2. Determinants and Matrix Inversion

Consider the system of n linear equations and n unknowns, $x_1 \dots x_n$.

$$y_1 = a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$y_n = a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n$$

This may or may not have a unique a solution. A unique solution will exist if and only if the determinant of the matrix $\boldsymbol{A}$, denoted $det\,\boldsymbol{A}$ or $|\boldsymbol{A}|$, is non-zero, i.e.

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

A definition of determinant of matrix $\boldsymbol{A}$ is

$$|\boldsymbol{A}| = a_{11}D_{11} - a_{12}D_{12} + a_{13}D_{13} - \cdots + (-1)^n a_{1n}D_n$$

where the D_{ij} are the determinants of the $(n-1)(n-1)$ sub-array formed by deleting the i^{th} row and j^{th} column from $\boldsymbol{A}$.

If $|\boldsymbol{A}| \neq 0$ and a unique solution exists, then this solution may be expressed

$$x_1 = b_{11}y_1 + \cdots + b_{1n}y_n$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$x_n = b_{n1}y_1 + \cdots + b_{nn}y_n$$

Expressing the two sets of equations in compact matrix form, $\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$ and $\boldsymbol{x} = \boldsymbol{B}\,\boldsymbol{y}$ so that on substitution $\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{B}\,\boldsymbol{y} = \boldsymbol{I}\,\boldsymbol{y}$. In other words

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

where $\boldsymbol{I}$ is the identity matrix. Thus it can be concluded that $\boldsymbol{A}$ and $\boldsymbol{B}$ are inverses such that

$\boldsymbol{B} = \boldsymbol{A}^{-1}$ and $\boldsymbol{A} = \boldsymbol{B}^{-1}$

If the determinant of a matrix is zero the matrix has no inverse, and it is said to be singular. If the determinant is non-zero its inverse does exist and the matrix is said to be non-singular.

There are numerous algorithms for computing the determinants and the inverses of matrices. However, a useful formula known as Cramers rule is as follows

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} adj\ (\boldsymbol{A})$$

Where the matrix denoted $adj\ (\boldsymbol{A})$ is known as the adjoint matrix. The i, j^{th} element of this matrix is the cofactor of a_{ji}. The cofactor or 'signed' minor is the determinant of the sub-matrix obtained by deleting the row and column containing a_{ij} and having a + sign if $i + j$ is even or a (-) sign if $i + j$ is odd.

For a 2×2 matrix the determinant and adjoint matrix are as follows

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad adj\ (\boldsymbol{A}) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$|\boldsymbol{A}| = a_{11}a_{22} - a_{21}a_{12}$$

For a 3×3 matrix the following must hold

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$adj\ \boldsymbol{A} = \begin{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} & \cdots & \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} & \cdots & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{bmatrix}$$

$$|\boldsymbol{A}| = a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

It can be shown that the determinant of the product of two $n \times n$ matrices is the product of the two determinants

$$|\boldsymbol{A}\,\boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|$$

It is also readily observed that $|\boldsymbol{I}| = 1$

Also $|\boldsymbol{A}^{-1}| = \frac{1}{|\boldsymbol{A}|}$

1.1.3. Transpose of a Matrix

If the rows of one matrix are the columns of another then the matrices are termed the transposes of each other, i.e. $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the transposes of each other.

Show (using examples) that (i) $(\boldsymbol{A}^T)^T = \boldsymbol{A}$

(ii) $(\boldsymbol{A}\,\boldsymbol{B})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

and if $\boldsymbol{A}$ is a square matrix that (iii) $|\boldsymbol{A}| = |\boldsymbol{A}^T|$

You may take for example

$$\boldsymbol{A} = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

1.1.4. Other Special Matrices

a) Symmetric Matrix

If $\boldsymbol{A} = \boldsymbol{A}^T$ then $\boldsymbol{A}$ is said to be symmetric. Replace the element in position a_{12} in $\boldsymbol{A}$ above with 2 (i.e. replace 7 with 2) and show that $\boldsymbol{A} = \boldsymbol{A}^T$.

b) Orthogonal Matrix

An orthogonal matrix is one whose transpose is equal to its inverse: $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$ so that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{A}^T$ Hence or otherwise show that the following matrix is not orthogonal.

$$\boldsymbol{A} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$$

c) Lower Triangular Matrix

A square matrix having all elements equal to zero in positions above the principal diagonal is called a lower triangular matrix. Its transpose is an upper triangular matrix. With reference to the general matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

show that the determinant of an upper and/or lower triangular matrix is the product of the diagnonal elements.

Deduce the condition under which upper and lower triangular matrices are guaranteed to be non-singular.

d) Diagonal Matrix

A special matrix which is both upper triangular and lower triangular is the diagonal matrix.

e.g. $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} = diag[\lambda_1, \lambda_2 \cdots \lambda_n]$

Show that $\mathbf{\Lambda} = \mathbf{\Lambda}^T$. Under what circumstances does $\mathbf{\Lambda} = \mathbf{\Lambda}^T = \mathbf{\Lambda}^{-1}$? i.e. symmetrical and orthogonal.

1.1.5. Notion of Rank and Linear Independence

The rank of a matrix $\boldsymbol{A}$ is denoted $rank\ (\boldsymbol{A})$ and is the largest sub matrix (which could include $\boldsymbol{A}$) which is nonsingular.

In other words it is the dimension of the array corresponding to the largest minor whose determinant is non-zero.

Alternatively it corresponds to the number of linearly independent rows (or columns) in the system matrix representation.

Exercises

* Determine the rank of the following matrices:

a) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$

c) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{pmatrix}$ d) $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 2 & 0 \end{pmatrix}$

■

When dealing with sets of simultaneous equations in n unknowns it is necessary to have n linearly independent equations.

$$a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n = y_1$$

$$a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n = y_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n = y_n$$

$$[\boldsymbol{A}][\boldsymbol{x}] = [\boldsymbol{y}]$$

** Note: In all the exercises, any solution obtained using Matlab should be checked by hand, and, where appropriate, solutions by hand should be checked using Matlab.*

To solve for the unknown vector $\boldsymbol{x}$ pre-multiply by $\boldsymbol{A}^{-1}$ to give $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{y}$.

A condition that $\boldsymbol{A}^{-1}$ exists must be fulfilled for a solution to exist. Equivalent conditions are:

i) $det\, \boldsymbol{A} \neq 0$
ii) $rank\, (\boldsymbol{A}) = n$

Both imply that there are n linearly independent equations. The rank of $\boldsymbol{A}$ is the dimension of the largest non-zero determinant.

1.2. Eigenvectors and Eigenvalues

1.2.1. Eigenvalues of a Square Matrix A

Consider an $n \times n$ square matrix $\boldsymbol{A}$**.** The determinant $|\boldsymbol{A} - \lambda \boldsymbol{I}|$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix is the characteristic polynomial of $\boldsymbol{A}$. It is a polynomial of order n in λ. The characteristic equation is given by $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$.

i.e.

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{13} \\ a_{21} & a_{22} - \lambda & \cdots & a_{23} \\ \vdots & \vdots & \ddots & \\ a_{31} & a_{32} & & a_{nn} - \lambda \end{bmatrix}$$

from which it can deduced that the characteristic equation may be given by

$$\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$$

The n roots (or zeros) of the characteristic eqaution are the eigenvalues of the matrix $\boldsymbol{A}$. They are also known as the characteristic roots, and, when dealing with dynamic systems, these are also known as the system poles. Note that the eigenvalues are either real or occur in complex conjugate pairs. In the special case of a triangular matrix (including by definition the diagonal matrix) the n diagonal elements are the n eigenvalues of the matrix.

1.2.2. Eigenvectors of a Square Matrix A

The eigenvectors corresponding to the eigenvalues λ_i are denoted $\boldsymbol{e}_i$ and satisfy the following equations

$$\boldsymbol{A}\, \boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i, \qquad i = 1 \dots n$$

Consider the case of n distinct eigenvalues and associated eigenvectors. It is now convenient to introduce the modal matrix of $\boldsymbol{A}$, which is defined as

$$\boldsymbol{M} = [\boldsymbol{e}_1 \vdots \boldsymbol{e}_2 \vdots \cdots \vdots \boldsymbol{e}_n]$$

i.e. the matrix whose columns are the eigenvectors of $\boldsymbol{A}$, and the spectral matrix, which is defined as

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

i.e. the diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$. Making use of $\boldsymbol{M}$ and $\boldsymbol{\Lambda}$ the n equations given by $\boldsymbol{A}\,\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$ may be expressed as

$$\boldsymbol{A}\,\boldsymbol{M} = \boldsymbol{M}\,\boldsymbol{\Lambda}$$

so that pre-multiplying by $\boldsymbol{M}^{-1}$ leads to

$$\boldsymbol{M}^{-1}\boldsymbol{A}\,\boldsymbol{M} = \boldsymbol{M}^{-1}\boldsymbol{M}\,\boldsymbol{\Lambda} = \boldsymbol{\Lambda}$$

i.e. when $\boldsymbol{A}$ has distinct eigenvalues it can be diagonalised by making use of the modal matrix.

1.2.3. The Importance of Eigenvectors

Consider the unforced dynamic system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x}$$

Make the substitution for the state vector

$$\boldsymbol{x} = \boldsymbol{M}\,\boldsymbol{z}$$

Where $\boldsymbol{M}$ is the modal matrix of $\boldsymbol{A}$**,** so that

$$\boldsymbol{M}\,\dot{\boldsymbol{z}} = \boldsymbol{A}\,\boldsymbol{M}\,\boldsymbol{z}$$

From which it follows that

$$\dot{\boldsymbol{z}} = \boldsymbol{M}^{-1}\,\boldsymbol{A}\,\boldsymbol{M}\,\boldsymbol{z} = \boldsymbol{\Lambda}\,\boldsymbol{z}$$

Due to the diagonal form of $\boldsymbol{\Lambda}$ the vector $\dot{\boldsymbol{z}}$ comprises uncoupled scalar equations

$$\dot{z}_i = \lambda_i z_i, \qquad i = 1 \ldots n$$

Which have the solution

$$\boldsymbol{z}_i(t) = \boldsymbol{z}_i(0)\, e^{\lambda_i t}, \qquad i = 1 \ldots n$$

where the $\boldsymbol{z}_i(0)$ are the initial values of the elements $\boldsymbol{z}_i$ (i.e. when $t = 0$).

The solution, therefore, to the original unforced state equation may be expressed as

$$\boldsymbol{x}(t) = \boldsymbol{M}\begin{bmatrix} \boldsymbol{z}_1(t) \\ \boldsymbol{z}_2(t) \\ \vdots \\ \boldsymbol{z}_n(t) \end{bmatrix} = \boldsymbol{M}\begin{bmatrix} e^{\lambda_1 t} & 0 & & \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ & & 0 & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} \boldsymbol{z}_1(0) \\ \boldsymbol{z}_2(0) \\ \vdots \\ \boldsymbol{z}_n(0) \end{bmatrix}$$

and making use of $\boldsymbol{z} = \boldsymbol{M}^{-1}\boldsymbol{x}$ [i.e. $\boldsymbol{z}(0) = \boldsymbol{M}^{-1}\boldsymbol{x}(0)$]

$$\boldsymbol{x}(t) = \boldsymbol{M}\, e^{\Lambda t}\, \boldsymbol{M}^{-1}\, \boldsymbol{x}(0)$$

in which $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}$

and $\boldsymbol{M}\, e^{\Lambda t}\, \boldsymbol{M}^{-1}$ is known as the state transition matrix, often denoted $\Phi(t)$ or $e^{\boldsymbol{A}t}$, so that the solution to the unforced system becomes

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\, \boldsymbol{x}(0)$$

Example

Consider the dynamic system described by the system matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

It is straightforward to compute the eigenvalues as

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2$$

and the corresponding eigenvectors as

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{and } \boldsymbol{e}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So that $\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

Straightforward computation of $\boldsymbol{M}^{-1}$ yields (which may be readily verified)

$$\boldsymbol{M}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

So that

$$e^{\boldsymbol{A}t} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

The result will be further expanded in Chapter 2 in the sections on Laplace transforms.

Exercises

i) Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$ corresponding to a second order dynamic system given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

Note that to obtain the eigenvalues you will need to solve the characteristic equation

$$det|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

ii) Obtain the eigenvectors corresponding to the eigenvalues.

i.e. solve $\lambda_i\, \boldsymbol{e}_i = \boldsymbol{A}\, \boldsymbol{e}_i,\qquad i = 1,2$

and form the modal matrix, $\boldsymbol{M} = [\boldsymbol{e}_1 \vdots \boldsymbol{e}_2]$

iii) Defining $\boldsymbol{e}_1$ to be the dominant eigenvector, show that the inverse of the modal matrix is given by

$$\boldsymbol{M}^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

iv) Using the modal matrix approach, obtain the solution to the unforced system i.e. obtain the state transition matrix $e^{\boldsymbol{A}t}$.

v) For the matrix $\boldsymbol{A}$ corresponding to the above second order dynamic system show that

$$\boldsymbol{A}^2 + 5\boldsymbol{A} + 6\boldsymbol{I} = 0$$

i.e. the matrix satisfies its own characteristic equation. Note that this is the Cayley Hamilton theorem. Hence or otherwise re-arrange for an expression for $\boldsymbol{A}^{-1}$ and compute its value. Check your result.

■

1.3. Dealing with Partitioned Matrices

When dealing with linear systems of vector-matrix equations, it is useful to consider partitioned matrices, e.g.

$$\begin{aligned}
\boldsymbol{y}_1 &= \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 + \cdots + \boldsymbol{A}_{1m}\boldsymbol{x}_m \\
\boldsymbol{y}_2 &= \boldsymbol{A}_{21}\boldsymbol{x}_1 + \boldsymbol{A}_{22}\boldsymbol{x}_2 + \cdots + \boldsymbol{A}_{2m}\boldsymbol{x}_m \\
\vdots &\qquad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\
\boldsymbol{y}_n &= \boldsymbol{A}_{n1}\boldsymbol{x}_1 + \boldsymbol{A}_{n2}\boldsymbol{x}_2 + \cdots + \boldsymbol{A}_{nm}\boldsymbol{x}_m
\end{aligned}$$

i.e. the $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ are vectors and the $\boldsymbol{A}_{ij}$ are matrices. In partitioned form

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \cdots \\ \boldsymbol{y}_2 \\ \cdots \\ \boldsymbol{y}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11} & \vdots & \boldsymbol{A}_{12} & & \vdots & \boldsymbol{A}_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \vdots & & & \vdots & \\ & \vdots & & & \vdots & \\ \boldsymbol{A}_{n1} & \vdots & & & \vdots & \boldsymbol{A}_{nm} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \cdots \\ \boldsymbol{x}_2 \\ \cdots \\ \boldsymbol{x}_m \end{bmatrix}$$

The $\boldsymbol{y}_i$, $\boldsymbol{x}_1$ are sub-vectors of a larger vector and the $\boldsymbol{A}_{ij}$ are sub-matrices of a larger matrix. The dashed lines are usually omitted for clarity and it is important to distinguish between the various entries. Consequently the following notation is adopted:

$\boldsymbol{A}$ Matrices are bold capital

$\boldsymbol{x}$ Vectors are bold lower case and are columns

$\boldsymbol{x}^T$ The transpose of a vector converts a column to a row

y, a Elements of vectors and matrices are non-bold case to indicate scalars

Matrices that are appropriately partitioned may be added, subtracted and multiplied. e.g.

$$\begin{bmatrix} \boldsymbol{A}_{11} & \vdots & \boldsymbol{A}_{12} \\ \cdots & \vdots & \cdots \\ \boldsymbol{A}_{21} & \vdots & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{11} & \vdots & \boldsymbol{B}_{12} \\ \cdots & \vdots & \cdots \\ \boldsymbol{B}_{21} & \vdots & \boldsymbol{B}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} & \vdots & \\ \cdots & \vdots & \cdots \\ & \vdots & \end{bmatrix}$$

is valid provided all matrix sums and products are of conformable dimension. Partitioning can be useful for obtaining inverses. Consider the system

$$\boldsymbol{y}_1 = \boldsymbol{A}\,\boldsymbol{x}_1 + \boldsymbol{B}\,\boldsymbol{x}_2$$

$$\boldsymbol{y}_2 = \boldsymbol{C}\,\boldsymbol{x}_1 + \boldsymbol{D}\,\boldsymbol{x}_2$$

or

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}$$

The inverse of

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

Can be expressed in terms of the inverses of the sub-matrices. Suppose sub-matrix $\boldsymbol{A}$ has an inverse, then solving for $\boldsymbol{x}_1$, leads to

$$\boldsymbol{x}_1 = \boldsymbol{A}^{-1}\boldsymbol{y}_1 - \boldsymbol{A}^{-1}\boldsymbol{B}\,\boldsymbol{x}_2$$

substituting into the equation for $\boldsymbol{y}_2$ yields

$$\boldsymbol{y}_2 = \boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{y}_1 - (\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})\boldsymbol{x}_2$$

Solving for $\boldsymbol{x}_2$

$$\boldsymbol{x}_2 = (\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{y}_1 - \boldsymbol{y}_2)$$

Substituting this into the solution for $\boldsymbol{x}_1$, yields

$$\boldsymbol{x}_1 = [\boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}\boldsymbol{C}\,\boldsymbol{A}^{-1}]\boldsymbol{y}_1 + \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}\boldsymbol{y}_2$$

The solution for $\boldsymbol{M}^{-1}$ has thus been formed

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}\boldsymbol{C}\,\boldsymbol{A}^{-1} & \vdots & \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1} \\ \cdots & \vdots & \cdots \\ (\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}\boldsymbol{C}\,\boldsymbol{A}^{-1} & \vdots & -(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1} \end{bmatrix}$$

The result is that the inverse of the large matrix $\boldsymbol{M}$ is expressed in terms of the smaller matrix inverses $\boldsymbol{A}^{-1}$ and $(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}$.

If $\boldsymbol{D}$ has an inverse the same procedure can be applied to obtain another expression for $\boldsymbol{M}^{-1}$, namely

$$\boldsymbol{M}^{-1} = \begin{bmatrix} -(\boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C} - \boldsymbol{A})^{-1} & \vdots & (\boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C} - \boldsymbol{A})^{-1}\boldsymbol{B}\,\boldsymbol{D}^{-1} \\ \cdots & \vdots & \cdots \\ \boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C} - \boldsymbol{A})^{-1} & \vdots & \boldsymbol{D}^{-1} - \boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C} - \boldsymbol{A})^{-1}\boldsymbol{B}\,\boldsymbol{D}^{-1} \end{bmatrix}$$

Since these two expressions are the same, each sub-matrix must be equal to the corresponding sub-matrix in both expressions.

In particular the following holds

$$-(\boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C} - \boldsymbol{A})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})^{-1}\boldsymbol{C}\,\boldsymbol{A}^{-1}$$

This is a version of the matrix inversion lemma which is attributed to Schur. The more familiar form is obtained by replacing $\boldsymbol{D}$ by $-\boldsymbol{D}$

$$(\boldsymbol{A} + \boldsymbol{B}\,\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{D})^{-1}\boldsymbol{C}\,\boldsymbol{A}^{-1}$$

The above is in the basis of many important results in system identification and control theory.

Exercises

The values of two variables $\boldsymbol{x}$ are related by a set of algebraic relations to two outputs $\boldsymbol{y}$. The outputs are measured and two further variables $\boldsymbol{z}$ are evaluated for subsequent use in control.

The vector $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is given by $\boldsymbol{y} = (\boldsymbol{A}\,\boldsymbol{B})\boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

$\boldsymbol{z}$ is related to $\boldsymbol{y}$ by $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \boldsymbol{C}\,\boldsymbol{y}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

Determine $\boldsymbol{D}$ in the expression $\boldsymbol{z} = \boldsymbol{D}\,\boldsymbol{x}$, hence obtain an explicit expression for $\boldsymbol{x}$ in terms of $\boldsymbol{z}$.

Determine the rank of the following square matrices:

(i) $\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 1 \\ 7 & 5 & 3 \\ 8 & 6 & 2 \end{bmatrix}$ (ii) $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 7 & 5 & 3 \\ 8 & 6 & 2 \end{bmatrix}$ (iii) $\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 1 & 3 \\ 5 & 2 & 9 \end{bmatrix}$ (iv) $\boldsymbol{A} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ -5 & -4 & -3 \end{bmatrix}$

Given that the rank of a rectangular matrix cannot exceed the lesser of the number of rows or the number of columns in a matrix. Determine the rank of the following:

(v) $\boldsymbol{A} = [1 \quad 2 \quad 3]$ (vi) $\boldsymbol{A} = \begin{bmatrix} 7 & 6 & 2 \\ 8 & 9 & 5 \end{bmatrix}$ (vii) $\boldsymbol{A} = [0 \quad 0 \quad 0]$

(viii) $\boldsymbol{A} = \begin{bmatrix} 0 & 4 & 2 & 6 \\ 9 & 1 & 3 & 2 \\ 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 8 \\ -1 & 3 & 4 & 2 \end{bmatrix}$

■

Chapter 2: Laplace Transforms and Transfer Function Representations

2.1. Laplace Transforms

The solution of differential equations involves finding the complementary function and the particular integral. Numerous methods exist. Here we shall consider the use of Laplace transforms. It allows consideration of discontinuous functions, very useful for dealing with practical problems in control engineering.

Definition

The Laplace transform $F(s)$ of a time function $f(t)$ is defined by the integral

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

The aim here is to provide practical examples of the use of Laplace transforms, and not to dwell on the mathematical rigour. It is assumed for all practical systems that the area under the graph of $f(t)$ remains finite and also that the following condition holds

$$\lim_{t\to\infty} f(t)e^{-st} = 0$$

where $s = \sigma + j\omega$ is a complex variable.

Procedure

To find the Laplace transform, denoted $\mathcal{L}\{f(t)\}$ of a function $f(t)$, multiply the function by e^{-st} and integrate the product $f(t)e^{-st}$, with respect to t over the interval $t = 0$ and $t = \infty$.

For each time function $f(t)$ which is Laplace transformable, the magnitude of the integrand $|f(t)|e^{-\sigma t}$ approaches zero as $t \to \infty$. The functions dealt with in practice are all Laplace transformable.

Examples

1) Find the Laplace transform of a constant i.e. $f(t) = a$ (constant)

$$\mathcal{L}\{a\} = \int_0^\infty ae^{-st}\,dt$$

$$= a\int_0^\infty e^{-st}\,dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{a}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{a}{s}(0-1)$$

$$\mathcal{L}\{a\} = \frac{a}{s}$$

2) Find the Laplace transform of $f(t) = e^{at}$ where a is a constant.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}(0-1)$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{(s-a)}$$

Note that the result is of fundamental importance when dealing with linear systems where the response is expressed as $e^{\lambda t}$ (recall from Chapter 1) where in general λ is the eigenvalue or pole of the dynamic system.

3) Find the Laplace transforms of $f(t) = \cos at$ and $f(t) \sin at$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \; dt$$

Recalling that $e^{j\theta} = \cos\theta + j\sin\theta$

i.e. cos θ is the real part of $e^{j\theta}$, so that $\cos at = \mathbb{R}(e^{jat})$

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\,\mathbb{R}\left(e^{jat}\right)dt$$

$$= \mathbb{R}\int_0^\infty e^{-(s-ja)t}dt$$

$$= \mathbb{R}\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right]_0^\infty$$

$$= \mathbb{R}\left[-\frac{1}{(s-ja)}[0-1]\right]$$

$$= \mathbb{R}\left(\frac{1}{(s-ja)}\right)$$

$$= \mathbb{R}\left(\frac{s+ja}{s^2+a^2}\right)$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Consequently from the same result it follows that, because $\sin\theta$ is the imaginary part,

$$\sin\ at = \mathbb{I}\left(\frac{s+ja}{s^2+a^2}\right)$$

$$\mathcal{L}\{\sin\ t\ \} = \mathbb{I}\frac{s+ja}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}$$

Exercises

Find the Laplace transforms of

i) $f(t)\sin 3t$, (ii) $f(t) = \cos 2t$, (iii) $f(t) = e^{4t}$

■

Dealing with constant factors

Consider $\mathcal{L}\{5\sin 2t\} = 5\mathcal{L}\{\sin 2t\}$

$$= 5\frac{2}{s^2+4}$$

$$= \frac{10}{s^2+4}$$

Dealing with several terms

Consider $\mathcal{L}\{e^{2t} + \sin 3t\} = \mathcal{L}\{e^{2t}\} + \mathcal{L}\{\sin 3t\}$

$$= \frac{1}{s-2} + \frac{3}{s^2+9}$$

Exercises

Find the Laplace transforms of

i) $f(t) = \{\sin 3t + \cos 4t\}$ ii) $f(t) = (6e^{-3t} - 6)$

iii) Find the Laplace transforms of $f(t) = \sinh at$ and $f(t) = \cosh at$

Use the following exponential definitions

$$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$$

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

show that $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$ and $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

■

Consider now $\mathcal{L}\{t^n\}$ where a) $n = 1$ and b) $n > 1$

a) Case of $n = 1$

$$\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st} = \left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= [0-0] + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s^2}[0-1]$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

b) Case of n as a positive integer > 1

Consider $\mathcal{L}\{t^n\} = \int_0^\infty t^n\, e^{-st} dt = \left[t^n \left(\frac{e^{-st}}{-s}\right)\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\, dt$

$$= -\frac{1}{s}[t^n e^{-st}]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

i.e. $I_n = \frac{n}{s}\, I_{n-1}$

which, continuing leads to

$$= \frac{n}{s}\,\frac{n-1}{s}\, I_{n-2}$$

$$= \frac{n}{s}\,\frac{n-1}{s}\,\frac{n-2}{s} I_{n-3}$$

so that

$$= \frac{n(n-1)(n-2)\ldots 3{,}2{,}1}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

e.g.

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

First shift theorem

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

i.e. whereas $\mathcal{L}\{f(t)\} = F(s)$, wherever s appears it is simply replaced by $s + a$

Example

Find the Laplace transform of $\mathcal{L}\{e^{-4t} t\}$.

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$ using the first shift theorem gives

$$\mathcal{L}\{e^{-4t} t\} = \frac{1}{(s+4)^2}$$

Exercises

Find the Laplace transforms of

i) $\mathcal{L}\{e^{-2t}\sin 3t\}$
ii) $\mathcal{L}\{e^{-3t}\cos 2t\}$
iii) $\mathcal{L}\{e^{-t}t^2\}$
iv) $\mathcal{L}\{4e^{2t}\sin t\}$
v) $\mathcal{L}\{e^{-t}\cosh 4t\}$
vi) $\mathcal{L}\{e^{-3t}(6+t^6)\}$

■

Useful theorem

If the Laplace transform of $f(t)$ is known, then the transforms of $tf(t)$ can be easily obtained.

If $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\{F(s)\}$

e.g.

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t\sin 2t\} = -\frac{\mathrm{d}}{\mathrm{ds}}\left\{\frac{2}{\mathrm{s}^2+4}\right\}$$

$$= -2\left\{\frac{-2s}{(s^2+4)^2}\right\}$$

$$= \frac{4s}{(s^2+4)^2}$$

Exercises

Show that

i)

$$\mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

ii)

$$\mathcal{L}\{t^2\sinh 2t\} = \frac{4(3s^2+4)}{(s^2-4)^3}$$

■

Summary of Results

Laplace transforms of common functions

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
t^n	$\frac{n!}{s^{n+1}}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Further Exercises

Find the Laplace transforms of the following

i) $\sin 4t$ (ii) $\cos 3t$ (iii) e^{2t} (iv) 3 (v) $5t^3$ (vi) e^{-t} (vii) $\sinh 2t$ (viii) $\cosh 5t$ (ix) $6 + e^{3t}$

■

Multiplying by t^n

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n - \frac{d^n}{ds^n}\{F(s)\}$

Exercises

Find the Laplace transforms for the following

i) $e^{2t}(t^2 - 5t + 6 + e^{-2t})$
ii) $e^{-2t}(3\ \cos 6t - 5\ \sin 6t)$
iii) $t^3 \cos t$
iv) $t^2 e^{3t}$
v) $e^{3t} \sin 5t$

■

2.2. Inverse Laplace Transforms

2.2.1. Inverse Laplace Transform

The inverse transform is denoted $\mathcal{L}^{-1}$. Basically recognising the form of the Laplace transform and the standard results already obtained it is possible to work backwards and forwards to find the inverse transform.

For example since $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Exercises

Evaluate the following inverse transforms

i) $\mathcal{L}^{-1}\left\{\frac{3}{s}\right\}$

ii) $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$

iii) $\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$

iv) $\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\frac{5}{s^2+9}\right\}$

■

2.2.2. Partial fractions

Finding the inverse transforms is straightforward when the results are recognisable, but when dealing with expressions such as

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{(s-2)(s-4)}\right\}$$

There is a need to re-express in a simpler form to obtain recognisable functions. To proceed use is made of partial fractions. The denominator is split into the two linear factors in this example to give

$$\frac{2s-6}{(s-2)(s-4)} = \frac{1}{(s-2)} + \frac{1}{(s-4)}$$

so that $f(t)$ is given by

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{(s-2)(s-4)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-4)}\right\}$$

$$= e^{2t} + e^{4t}$$

Rules of partial fractions

1. Numerator of lower degree than denominator (otherwise first derivative divides out)

2. Factorise the denominator into its prime factors

3. A linear factor $(s + a)$ gives a partial fraction $\frac{A}{(s+a)}$ where A is a constant

4. Repeated factors $(s + a)^2$ gives $\frac{A}{(s+a)} + \frac{B}{(s+a)^2}$

5. Similarly $(s + a)^3$ gives $\frac{A}{(s+a)} + \frac{B}{(s+a)^2} + \frac{B}{(s+a)^3}$

6. A quadratic factor gives $\frac{Ps+Q}{s^2+ps+q}$

7. $(s^2 + ps + q)^2$ gives $\frac{Ps+Q}{s^2+ps+q} + \frac{Rs+T}{(s^2+ps+q)^2}$

2.3. Properties of the Laplace Transform

2.3.1. Laplace Transforms of Derivatives

This is a useful property because it allows the values of signals at time $t = 0$ to be taken into account, thereby enabling initial conditions to be taken into consideration.

Consider the Laplace transform of the differential $\frac{dx}{dt}$ so that from the definition of the Laplace transform

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = \int_0^\infty \frac{dx}{dt} e^{-st}\ dt$$

Integrating by parts yields

$$= [x(t)e^{-st}]_0^\infty + s\int_0^\infty x(t)e^{-st}\ dt$$

The integral in the second term represents $X(s)$, the Laplace transform of $x(t)$. Inserting the limits into the first term gives

$$[x(t)e^{-st}]_0^\infty = \lim_{t\to\infty} x(t)e^{-st} - x(0)$$

The first term becomes zero, since $x(t)$ is Laplace transformable, so that

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

where $X(s)$ is the Laplace transform of $x(t)$ and $x(0)$ is the value of $x(t)$ when $t = 0$.

The above is a useful result, such that repeated application leads to the general expression for the Laplace transform of an n^{th} order differential equation

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}x^1(0) \dots - x^{n-1}(0)$$

where the notation $x^i(0)$ denotes the i^{th} derivative of $x(t)$ at $t = 0$.

2.3.2. Initial and Final Value Theorems

Initial Value Theorem

Consider the time differential property

$$\int_0^\infty x(t)e^{-st}\ dt = sX(s) - x(0)$$

Taking the limit on both sides as $s \to \infty$ and noting that $\lim_{s\to\infty} e^{-st} = 0$, it may be deduced

$$0 = \lim_{s\to\infty} sX(s) - x(0)$$

so that

$$x(0) = \lim_{s\to\infty} sX(s)$$

Final Value Theorem

Consider again the time differential property

$$\int_0^\infty x(t)e^{-st}\ dt = sX(s) - x(0)$$

Taking the limit on both sides as $s \to 0$

$$[x(t)]_0^\infty = \lim_{s\to 0} sX(s) - x(0)$$

$$x(\infty) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

so that

$$x(\infty) = \lim_{s\to 0} sX(s)$$

2.4. Differential Equation and Transfer Function Representations

2.4.1. Differential Equation

Continuous systems are generally described by differential equations of the form

$$\alpha_n \frac{d^n y}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \alpha_{n-2} \frac{d^{n-2} y}{dt^{n-2}} + \cdots + \alpha_0 y$$

$$= \beta_m \frac{d^m u}{dt^m} + \beta_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + \beta_0 u$$

By making use of the differential property of the Laplace transform and by assuming the initial conditions are all zero, $\frac{d^n y}{dt^n}$ can be replaced by $s^n Y(s)$, $\frac{d^{n-1} y}{dt^{n-1}}$ can be replaced by $s^{n-1} Y(s)$, and so on. Repeated use of this procedure leads to

$$(\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0) Y(s) = (\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_0) U(s)$$

Rearranging gives

$$\frac{Y(s)}{U(s)} = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0}$$

The ratio of the Laplace transform of the output signal and the input signal is known as the system transfer function. In all practical systems the order of the numerator is less than the denominator, i.e. $m < n$.

2.4.2. Transfer Function

The ratio $\frac{Y(s)}{U(s)}$ is often denoted $G(s)$, and referred to as a transfer function, i.e. $G(s) = \frac{Y(s)}{U(s)}$.

In many engineering problems the ability to handle 1st and 2nd order systems is of importance, noting that the zeros of both the numerator and denominator polynomials in s either occur as real or complex conjugate pairs. It is common practice to denote the numerator and denominator polynomials as $B(s)$ and $A(s)$, respectively, so that

$$G(s) = \frac{B(s)}{A(s)}$$

and

$$Y(s) = G(s)U(s) = \frac{B(s)}{A(s)} U(s)$$

The zeros of the numerator polynomial, i.e. the solutions of

$$B(s) = 0$$

are termed the system zeros. The zeros of the denominator polynomial, i.e. the solutions of

$$A(s) = 0$$

are termed the system poles. Recall that the above is also known as the characteristic equation and the poles of the system are also the eigenvalues of the corresponding system matrix $\boldsymbol{A}$, which is obtained when expressing the system in a state space form $\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{b}\,u$, i.e. noting that $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ is equivalent to $A(s)$ when s is replaced by λ.

2.4.3. Alternative State Space Representations

When considering continuous systems, it has been shown that there is a unique mapping between the differential equation and transfer function representation. Noting the assumptions made regarding initial conditions. Essentially the state space representation, which is briefly introduced here, provides an alternative form in which the n^{th} order differential equation is expressed as n 1st order differential equations. However the important point to note is that the state space representations are not unique.

Having introduced the Laplace transform its inverse and considered the use of partial fractions as well as examining some key properties, consideration is again given to the eigenvalue/eigenvector issues that arise in a state space system representation. However, in contrast to the time domain considerations of Chapter 1, attention is now focused towards functions of a complex variable $s = \sigma + j\omega$ via the Laplace transform. Recall for a general continuous derivative

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

Consider now the Laplace transform of a linear system expressed in state-space form. The general form of the state space representation for a single input single output (SISO) system is

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{b}\,u$$

$$y = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

Depending on the application, there are various forms for the above representation and each will depend on the definition of the system state vector denoted $\boldsymbol{x}$. For a general SISO system the system matrix is an $n \times n$ square matrix denoted $\boldsymbol{A}$, and the $n \times 1$ input and $1 \times n$ output vectors are denoted here as $\boldsymbol{b}$ and $\boldsymbol{c}^{\mathrm{T}}$, respectively.

Application of the Laplace transform to the state equation, analogously with the scalar case, following the steps, denoting the transforms of the vectors by uppercase letters yields

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{A}\,\boldsymbol{X}(s) + \boldsymbol{b}\,U(s); \qquad Y(s) = \boldsymbol{c}^{\mathrm{T}}\,\boldsymbol{X}(s)$$

Rearranging leads to

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{x}(0) + \boldsymbol{b}\,U(s)$$

so that the Laplace transform of the state vector $\boldsymbol{x}$ becomes

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}\,U(s)$$

i.e. the sum of two components, one dependent on initial conditions, and the other on the inputs.

Recalling from Chapter 1 that $\boldsymbol{A}^{-1} = \frac{adj\,\boldsymbol{A}}{det\,\boldsymbol{A}}$ gives

$$\boldsymbol{X}(s) = \frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{det\,(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{x}(0) + \frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{det\,(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{b}\,U(s)$$

so that the output $Y(s)$ may be expressed

$$Y(s) = \boldsymbol{c}^{\mathrm{T}}\frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{det\,(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{x}(0) + \boldsymbol{c}^{\mathrm{T}}\frac{adj\,(s\boldsymbol{I} - \boldsymbol{A})}{det\,(s\boldsymbol{I} - \boldsymbol{A})}\boldsymbol{b}\,U(s)$$

Recall from Chapter 1 the denominator as the characteristic polynomial in the Laplace variable s.

Further links are also apparent (see Chapter 1) between the inverse Laplace transform of the resolvent matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ and the state transition matrix $e^{\boldsymbol{A}t}$.

2.4.4. System stability and pole location (continuous-time)

Clearly for stability the real parts of the complex poles must be negative. When considering the complex s-plane

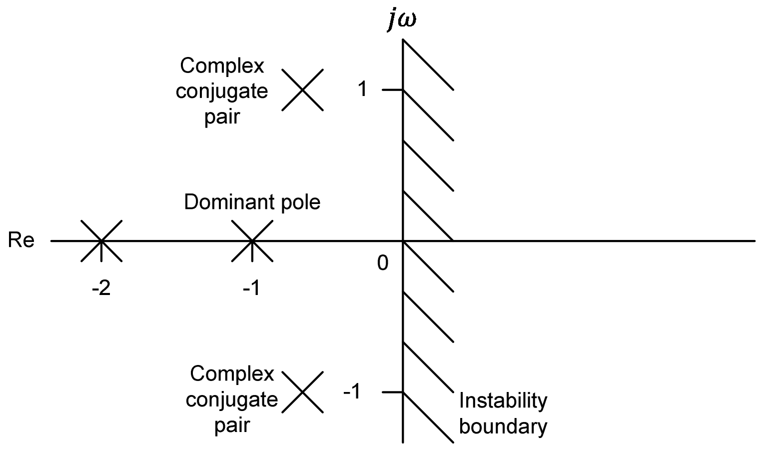


Figure 3.5 The complex s-plane

The zeros of the *nth* order characteristic equation in λ

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

or equivalently the nth order denominator of the transfer function in s or equivalently the eigenvalues of the system matrix $\boldsymbol{A}$ must have negative real part to ensure system stability. If the real part is zero the stability is marginal and if the real parts are positive the system will be unstable.

Exercises

Find the partial fractions of

1. $\frac{s+5}{(s-3)(s+2)}$
2. $\frac{s^2}{(s-2)(s+3)^2}$
3. $\frac{2s-8}{(s^2-8s+15)}$
4. $\frac{5s-8}{s(s-4)}$
5. $\frac{(s^2-2s+3)}{(s-2)^2}$

Hence determine the inverse Laplace transforms of the above transforms.

Determine the Laplace transforms of

6. $t \cos at$
7. $t^2 e^{3t}$
8. $e^{-2t}(3\, cos\, 6t - 5\, sin\, 6t$
9. $sinh\, 2t - sin\, 2t$
10. $e^{-4t}\, cosh\, 2t$
11. $cosh^2\, 4t$
12. $e^{-2t}\, sin\, \omega t$

Determine the inverse Laplace transforms of

13. $\frac{2s-1}{s^2+4s+29}$
14. $\frac{2s+4}{(s^2+4s+5)^2}$
15. $\frac{s-1}{(s+3)(s^2+2s+2)}$
16. $\frac{s^2}{(s^2+1)(s^2+2)}$
17. $\frac{2s^2+s-10}{(s-4)(s^2+2s+2)}$

■

Chapter 3: Feedback control systems and discretisation

3.1. Feedback systems

Systems considered so far have been open-loop systems, i.e. without feedback. In practice systems are normally configured using feedback leading to a closed-loop system which has desired properties. As has been highlighted in Chapters 1 and 2 The fundamental properties of a system are the eigenvalues/eigenvectors which characterise the system, equivalently the characteristic equation. In this chapter attention is focused on the characteristic equation of the closed-loop system, i.e. the system with feedback.

A simple error driven unity feedback system with a static controller gain K is shown in Figure 3.1, where $G(s)$ denotes the transfer function of the open-loop system.

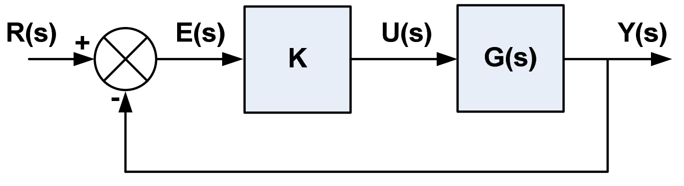


Figure 3.1 Closed-loop control

The quantities $R(s), E(s), U(s)$ and $Y(s)$ are the Laplace transforms of the signals corresponding to the time functions $r(t), e(t), u(t)$ and $y(t)$, respectively. Using straightforward block diagram algebra

$$Y(s) = G(s)\,U(s)$$

$$U(s) = K\,E(s)$$

$$E(s) = R(s) - Y(s)$$

So that the closed loop transfer function in the Laplace variable s may be expressed as

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)} = G_{CL}(s)$$

Recalling from Chapter 2 that $G(s) = \frac{B(s)}{A(s)}$, it follows that

$$G_{CL}(s) = \frac{KB(s)}{A(s) + KB(s)}$$

so that the characteristic equation of the closed-loop system becomes

$$A(s) + KB(s) = 0$$

which implies that the poles of the closed-loop feedback system, hence the equivalent eigenvalues/eigenvectors have been changed via feedback. The closed-loop system now behaves in a manner which has been designed via feedback, and the controller gain K can be adjusted to produce a desired closed-loop dynamic response.

It is also important to consider the steady-state gain of the closed loop system. This can be found using the final value theorem, i.e. setting $s \longrightarrow 0$

Steady-state gain (SSG)

$$SSG = \lim_{s \longrightarrow 0} G_{CL}(s) = \left. \frac{KB(s)}{A(s) + KB(s)} \right|_{s=0}$$

i.e.

$$SSG = \frac{K\beta_0}{\alpha_0 + K\beta_0}$$

It is clear from the above that unless $\alpha_0 = 0$ (implying a type 1 system, i.e. a system where a factor s can be taken out as a common factor from the denominator) the steady-state gain will not be unity, but will approach unity as $K \to \infty$. Unfortunately, a consequence of increasing K, depending on the system, is that of increased oscillation at the system output and possible instability. Alternative forms of control other than the simple static gain K will overcome these problems. Whilst the scope is wide and not intended for such a text, brief consideration will be given to the standard industrial three-term proportional + integral + derivative (PID) controller as well as an introduction to state variable feedback control.

The common structures for feedforward and combined feedforward/feedback compensated control systems are illustrated diagrammatically as follows (note that these use unity feedback in their outer loop)

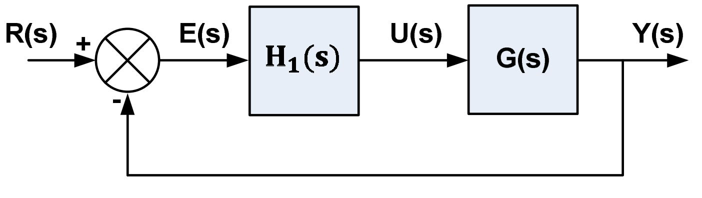


Figure 3.2 Closed-loop feedforward control

Figure 3.2 shows a feedback control system employing a feedforward compensator ($H_1(s)$ denotes feedforward compensator c.f. to K in previous example).

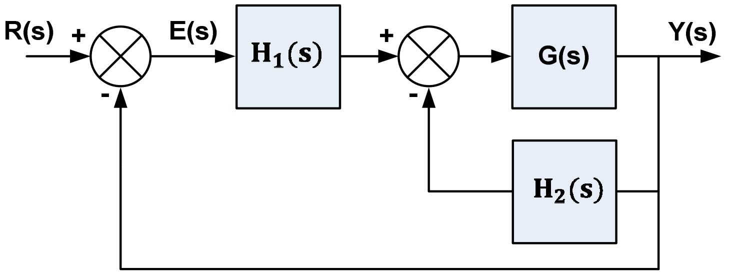


Figure 3.3 Closed loop feedforward/feedback control

Figure 3.3 shows a feedback control system employing feedforward/feedback compensator ($H_1(\mathrm{s})$, $H_2(\mathrm{s})$ denote feedforward and feedback compensator, respectively).

Exercises

Show that the block diagrams of the above closed loop systems are, respectively

i) $\frac{H_1 G}{1+H_1 G}$ and ii) $\frac{H_1 G}{(1+H_1)G+H_2}$

■

3.2. Standard three-term PID control

A common form of control used in practice is the industry standard three-term proportional + integral + derivative (PID) control scheme. The three terms act on the error signal to multiply, integrate over time and differentiate, respectively (note that in the simple static gain controller case, K was the proportional gain term). The PID controller is configured as follows

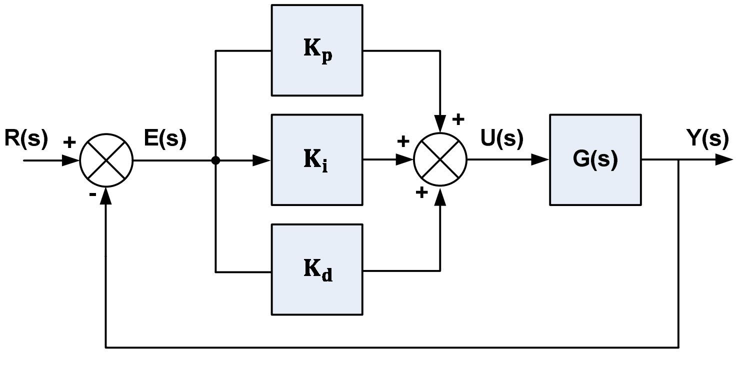


Figure 3.4 PID controller

Note that when the instantenous error signal is zero the proportional and derivative actions are also zero, so the only contribution to the control action is that of the integral term. This will build up over time as the error signal is integrated to provide the steady-state control action to hold the system output steady at steady-state.

A more common configuration for the controller is the structure

$$\frac{U(s)}{E(s)} = K_p(1 + \frac{K_i}{s} + K_d s)$$

so that K_p affects all three terms of the controller. It is also common to replace K_i and K_d with $\frac{1}{T_i}$ and T_d, respectively, where T_i and T_d denote the integral time and the derivative time constants, which are more widely used by engineers in the field, to give

$$\frac{U(s)}{E(s)} = K_p(1 + \frac{1}{T_i s} + T_d s)$$

Note that this can be expressed as

$$= \frac{K_p}{T_i} \frac{(T_i s + 1 + T_i T_d s^2)}{s}$$

i.e. the PID controller introduces a pole at the origin (from denominator) and two zeros (from numerator). These will allow the control engineer to ‘shape’ the response of the closed loop system. In practice the PID controller is implemented in the discrete-time form, see Section 3.4.

3.3. State-space representations and state variable feedback

The continuous-time state space representation for a single-input single-output (SISO) system is given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$$

$$y = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$$

where the original n^{th} order differential equation is expressed as n 1st order differential equations. As stated in Chapter 2, the state space representation is non-unique and depends on the user choice for the state vector. There are, however, certain forms which are useful in control. One particular form, known as the phase variable canonical form, is given, where the state vector is defined such that

$$x_1 = y$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = x_3$$

$$\vdots$$

$$\dot{x}_n = \quad = x_{n-1}$$

Illustrative example

Consider a second order dynamical system expressed in the differential equation form

$$\frac{d^2y}{dt^2} + \frac{5dy}{dt} + 6y = 12u$$

or

$$\ddot{y} + 5\dot{y} + 6y = 12u$$

Equivalently in transfer function form

$$G(s) = \frac{12}{s^2 + 5s + 6} = \frac{12}{(s+2)(s+3)}$$

From which it is clear that the system poles are -2 and -3 i.e. the values of s for which $A(s) = 0$. Returning to the state space representation and noting that the two 1st order differential equations are given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -6x_1 - 5x_2 + 12u$$

The state space matrix and vectors may be stated as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that in the phase variable form the last row of the matrix $\boldsymbol{A}$ contains the coefficients of the transfer function denominator in reverse order with sign change. This is always the case when defining the state vector as indicated above. Note the forms of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$**,** which also follow from the definition of the state vector.

Defining now the state variable feedback control law

$$u = \boldsymbol{f}^T \boldsymbol{x}$$

where $\boldsymbol{f}$ is a feedback vector which is derived in order to achieve a desired closed-loop system response. Substitute the state variable feedback control into the state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{f}^{\boldsymbol{T}}\boldsymbol{x}$$

$$= [\boldsymbol{A} + \boldsymbol{b}\boldsymbol{f}^{\boldsymbol{T}}]\boldsymbol{x}$$

where $\boldsymbol{A}_{\boldsymbol{CL}} = [\boldsymbol{A} + \boldsymbol{b}\boldsymbol{f}^{\boldsymbol{T}}]$ is the desired closed-loop system matrix, having desired eigenvalues (for example to achieve a certain response in terms of damping factor.

Note that $\boldsymbol{A}_{\boldsymbol{CL}} = [\boldsymbol{A} + \boldsymbol{b}\boldsymbol{f}^{\boldsymbol{T}}]$ is a matrix equality which is satisfied if and only if the corresponding elements are equal. In general this will involve *n* equations and *n* unknowns. However, for a solution to exist the *n* equations must be linearly independent i.e. of full rank. This condition was introduced in Chapter 1, and is now to be re-visited to introduce two key concepts in state variable feedback control, namely controllability and observability.

Note that if a system is mathematically uncontrollable there is no point in attempting to design a control system. Similarly for a system which is unobservable, it will not be possible to obtain any feedback nor will it be possible to observe any control effects.

3.3.1 Controllability and Observability (Kalman matrices)

A system is said to be controllable if all of the system states can be driven from some initial state to some final state in a finite time (loosely, 'looking inside' the system, the states can be manipulated from the input). Mathematically this can be expressed in terms of the Kalman controllability test matrix, defined as

$$\boldsymbol{K}_{\boldsymbol{c}} = [\boldsymbol{A}^{\boldsymbol{n}-\boldsymbol{1}}\boldsymbol{b} : \boldsymbol{A}^{\boldsymbol{n}-\boldsymbol{2}}\boldsymbol{b} : \cdots : \boldsymbol{A}\boldsymbol{b} : \boldsymbol{b}]$$

A system is controllable if and only if the matrix $\boldsymbol{K}_{\boldsymbol{c}}$ has full rank, i.e. $rank\ (\boldsymbol{K}_{\boldsymbol{c}}) = n$.

A system is said to be state observable if all of the system states are available for feedback and the output is affected by all states (loosely 'looking inside' the system from the output all the states affect the response). Mathematically, this can be expressed in terms of the Kalman Observability test matrix, defined as

$$\boldsymbol{K}_{\boldsymbol{o}} = \begin{bmatrix} \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{n}-\boldsymbol{1}} \\ \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{n}-\boldsymbol{2}} \\ \vdots \\ \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{A} \\ \boldsymbol{c}^{\boldsymbol{T}} \end{bmatrix}$$

A system is state observable if and only if the matrix $\boldsymbol{K}_{\boldsymbol{o}}$ has full rank, i.e. $rank\ (\boldsymbol{K}_{\boldsymbol{o}}) = n$. For the example system $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$ $\boldsymbol{c}^{\boldsymbol{T}} = [1 \quad 0]$

$\boldsymbol{K_c} = \begin{bmatrix} 12 & 0 \\ -60 & 12 \end{bmatrix}$ full rank 2, equivalently $|\boldsymbol{K_c}| \neq 0, |\boldsymbol{K_c}| = 204$

$\boldsymbol{K_o} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ full rank 2, $|\boldsymbol{K_o}| = 1$

3.3.2 Controllability and Observability (Modal matrix)

Recall from Chapter 1 the diagonalisation of the system matrix $\boldsymbol{A}$ to the spectral matrix $\boldsymbol{\Lambda}$ comprising of eigenvalues on the diagonal.

Make the transformation $\boldsymbol{x} = \boldsymbol{Mz}$, so that $\dot{\boldsymbol{x}} = \boldsymbol{M\dot{z}},$ yielding

$$\boldsymbol{M}\dot{z} = \boldsymbol{AMz} + \boldsymbol{b}u$$

$$y = \boldsymbol{c^T M z}$$

pre-multiply by $\boldsymbol{M^{-1}}$

$$\dot{\boldsymbol{z}} = \boldsymbol{M^{-1}AM}z + \boldsymbol{M^{-1}b}u$$

Note that the spectral properties are retained through such a linear transformation

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda z} + \boldsymbol{M^{-1}b}u$$

$$y = \boldsymbol{c^T M z}$$

Since $\boldsymbol{\Lambda}$ is a diagonal matrix the only condition for controllability is for the vector $\boldsymbol{M^{-1}b}$ to have no zero elements. Similarly the output to vector $\boldsymbol{c^T M}$ should have no zero elements for observability. Consider the system comprising $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c^T}$ as follows

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 12 \end{bmatrix} \text{ and } \boldsymbol{c^T} = [1 \quad 0]$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -3$ and corresponding eigenvectors are $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \boldsymbol{e_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

so that $\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$ and $\boldsymbol{M^{-1}} = \frac{1}{-1}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$

Controllability: $\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 12 \\ -12 \end{bmatrix}$ no zero elements so controllable as expected

Observability: $[1 \quad 0]\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} = [1 \quad 1]$ no zero elements so observable as expected

3.3.3 State variable feedback

Consider the state variable feedback control law

$$u = \boldsymbol{f}^{\boldsymbol{T}}\boldsymbol{x}$$

There are many methods of obtaining the feedback vector $\boldsymbol{f}$, and mostly involve either equating coefficients of equivalent matrices or polynomials or eigenvalues. Consider again the example system, having $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^{\boldsymbol{T}}$, as follows

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 12 \end{bmatrix} \text{ and } \boldsymbol{c}^{\boldsymbol{T}} = [1 \quad 0]$$

with eigenvalues at $\lambda_1 = -2,\ \lambda_2 = -3$. Suppose, arbitrarily, that the pole at -2 is to be changed to -4. The objective is to find the feedback vector to achieve this. The desired closed loop system matrix $\boldsymbol{A}_{\boldsymbol{CL}}$, may be expressed as

$$\boldsymbol{A}_{\boldsymbol{CL}} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

Recall that the elements relate to the desired closed-loop transfer function, with these relating to product and sum of the constants in the linear factors, so that

$$\boldsymbol{A}_{\boldsymbol{CL}} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} + \begin{bmatrix} 0 \\ 12 \end{bmatrix} [f_1 \quad f_2] \qquad \begin{bmatrix} 0 & 0 \\ 12f_1 & 12f_2 \end{bmatrix}$$

which yields two equations in two unknowns

$$-12 = -6 + 12f_1 \ \rightarrow \ f_1 = -{}^{6}/_{12} = -{}^{1}/_{2}$$

$$-7 = -5 + 12f_2 \ \rightarrow \ f_2 = -{}^{2}/_{12} = -{}^{1}/_{6}$$

3.3.3.1. Realisations of the state variable feedback

In many practical cases the state vector may not be directly measurable (but is observable). This means that it can be reconstructed using state estimation or state observer algorithms. Diagrammatic realisations of the state-space systems including their closed-loop feedback configurations are given in Figures 3.5 and 3.6.

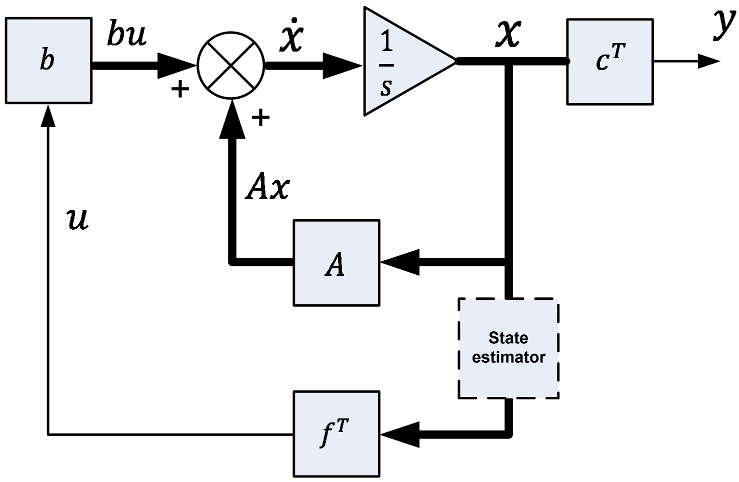


Figure 3.5 State variable feedback implementation

Lines in bold represent vectors with the remainder denoting scalar values.

Returning to the phase variable canonical form where

$x_1 = y$ (definition) $x_2 = \dot{x}_1 \rightarrow \ddot{x}_2 = \dot{x}_1$

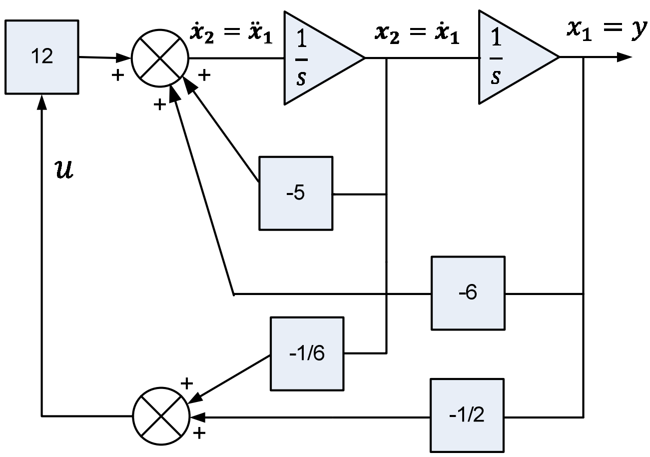


Figure 3.6 Phase variable canonical form of closed-loop system

3.4. Discrete time computer control

Most control systems are implemented digitally utilising microprocessor technology. The system being controlled is continuous, $G(s)$ but the controller is discrete. However, in order to design a discrete-time model based controller a discrete-time model of the plant is required.

In the design of such computer control systems a zero order hold (ZOH) is required to be modelled and accomodated together with the linear transfer function of the plant. This is required for design purposes because the inclusion of the ZOH will affect the closed-loop dynamic response.

The z-form transfer function, denoted $G(z)$, for the combined ZOH and the plant $G(s)$, may be obtained from

$$G(z) = \frac{(z-1)}{z} \ Z \ \left\{\frac{G(s)}{s}\right\}$$

where the continuous-time transfer function of the ZOH is given by

$$G_{OH}(s) = \frac{1-e^{-sT_s}}{s}$$

which may be interpreted as the application of a step followed by the removal of a step after a delay of T_s second, where T_s is the sampling interval. Thus each individual 'step' in the staircase input signal can be thought of in this way i.e. application followed by removal of a step at each successive sample step.

Having obtained $G(z)$ for the combined ZOH and plant it is then possible to design a discrete-time control system having a discrete-time transfer function, denoted $D(z)$. It is important to note that $D(z)$ does not incorporate the ZOH, as this has already been taken into account in $G(z)$. Note that discretisation in Matlab automatically defaults to use the ZOH approach unless otherwise stated.

3.4.1. Methods of discretising a continuous control algorithm (e.g. PI/PID)

Three discretisation methods are now described which may be used to convert existing continuous-time control systems into their discrete-time counterparts. Most approaches are based on the idea that algebraic multiplication by s implies differentiation in the time domain. Making use of the simple Euler approximation to the derivative of the gradient of the time response, for example the continuous-time error signal, denoted $e(t)$, sampled at the present time instant k, is given by:

$$\left.\frac{de}{dt}\right|_k = \frac{\text{current sample} - \text{previous sample}}{\text{sampling interval}} = \frac{e_k - e_{k-1}}{T_s} \approx se|_k$$

In the discrete-time domain this can be expressed as

$$se \approx \frac{e_k(1-z^{-1})}{T_s}$$

where z^{-1} is the backward time shift operator.

A more accurate approach uses the Tustin transformation

$$s = \frac{2}{T_s}\frac{(z-1)}{(z+1)}$$

which is more usefully implemented as

$$s = \frac{2}{T_s}\frac{(1-z^{-1})}{(1+z^{-1})}$$

because past values of a signal and are always available in practice.

The matched pole-zero method exploits the property that a pole in the continuous s-plane maps to a pole in the discrete z-plane using the expression

$$(\text{pole in z}) = e^{(\text{pole in s})T_s}$$

or

$$(\text{pole in s}) = T_s^{-1}\ln{(\text{pole in z})}$$

In this approach each linear factor $(s+a)$ is replaced by $(z-e^{aT_s})$, and a steady-state gain term is required to be deduced, since $G(s)|_{s=0} \equiv G(z)|_{z=1}$.

Example

Consider the continuous transfer function which could be thought of as a controller/compensator in the s-plane

$$D(s) = \frac{1.5(s+1)}{(s+3)}$$

Prior to discretising it is useful to note some of the properties. Setting $s \to 0$ in steady-state reveals a gain of 0.5. There is a pole at -3 and a zero at -1 in the s-plane. The objective here is to discretise using the above three methods and compare the resulting discrete-time algorithms. Choose $T_s = 0.1\text{s}$

Simple Euler method $s = (1-z^{-1})/T_s$

Replace s with $(1-z^{-1})/T_s$

Setting $T_s = 0.1 \to s \simeq 10(1-z^{-1})$, so that

$$D_1(z) = \frac{1.5(10(1-z^{-1})+1)}{(10(1-z^{-1})+3)}$$

$$= \frac{1.5(11 - 10z^{-1})}{(13 - 10z^{-1})}$$

$$= \frac{(16.5 - 15z^{-1})}{(13 - 10z^{-1})}$$

dividing through by 13 to give a leading value of 1 in the denominator leads to

$$D_1(z) = \frac{1.2692 - 1.1538\, z^{-1}}{1 - 0.7692\, z^{-1}}$$

Knowing that

$$D_1(z) = \frac{U(z)}{E(z)} \quad \left\{\frac{\text{controller output}}{\text{error}}\right\}$$

$$u_k - 0.7692\, u_{k-1} = 1.2692\, e_k - 1.1538\, e_{k-1}$$

so that

$$u_k = 1.2692\, e_k - 1.1538\, e_{k-1} + 0.7692\, u_{k-1}$$

becomes the algorithm coded into the microcontroller.

Tustin method $\frac{2}{T_s}\frac{(1-z^{-1})}{(1+z^{-1})}$

$$D_2(z) = \frac{1.5\left(\frac{2}{T_s}\frac{(1-z^{-1})}{(1+z^{-1})} + 1\right)}{\left(\frac{2}{T_s}\frac{(1-z^{-1})}{(1+z^{-1})} + 3\right)}$$

Multiplying top and bottom by $(1 + \mathrm{z}^{-1})$ and setting $T_s = 0.1$

$$= \frac{1.5\big(20(1 - z^{-1}) + (1 + z^{-1})\big)}{20\big((1 - z^{-1}) + 3(1 + z^{-1})\big)}$$

$$= \frac{1.5(21 - 19z^{-1})}{23 - 17z^{-1}}$$

dividing through by 23 to obtain the leading 1 in the denominator

$$D_2(z) = \frac{1.3696 - 1.2391\, z^{-1}}{1 - 0.7391\, z^{-1}}$$

so that the corresponding algorithm becomes

$$u_k = 1.3696 e_k - 1.2391\, e_{k-1} + 0.7391 u_{k-1}$$

Matched pole-zero method

Numerator term $e^{-1(0.1)} = 0.9048$

Denominator term $e^{-3(0.1)} = 0.7408$

$$D_3(z) = A\frac{(z - 0.9048)}{(z - 0.7408)}$$

Setting $z = 1$ for steady state and knowing that the steady gain is equal to 0.5 allows the determination of the constant A, i.e.

$$D_3(z) = \frac{1.3613\,(1 - 0.9048\,z^{-1})}{(1 - 0.7408\,z^{-1})}$$

so that

$$u_k = 1.3613e_k - 1.2317\,e_{k-1} + 0.7408\,u_{k-1}$$

becomes the algorithm.

Exercises

1. Determine the z-form function of the following

$$\frac{10(s+1)}{s(s+5)}$$

using the simple approximation $s \simeq (1 - z^{-1})/T$, assume that T = 0.1seconds, hence or otherwise write down the corresponding difference equation.

2. Repeat using the Tustin approximation and the matched pole-zero method. Comment on the result obtained.

3. Repeat using the zero order hold approach and explain why this is used in controller design. Compare the all the responses when applying a unit step.

4. For the general system given by

$$G(s) = \frac{k}{(s+a)s}$$

obtain a general form for $G(z)$ via the zero order hold approach in terms of k, a and T.

5. For the z-form function obtained above in (4) select sampling intervals T = 0.01, 0.1 and 1.0. Evaluate the poles and zeros of the system and comment on the choice of T.

■

3.4.2. Discretisation of a PID controller

Consider the following form of a PID

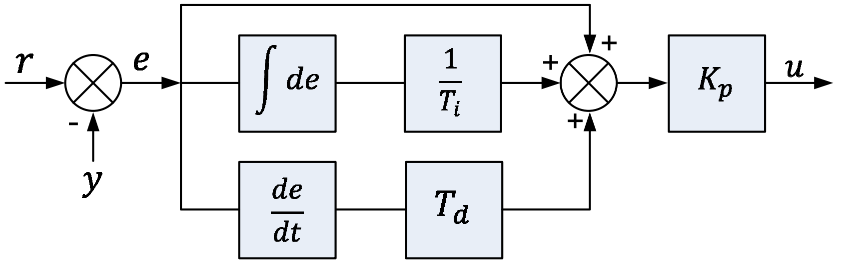


Figure 3.7 Alternative form of a PID controller

The transfer function may be expressed as

$$D(s) = K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$$

Again any of the three methods may be applied to the derivative term. However, the integral term requires some thought. The time integral of the error signal is as the name implies is the area under the error signal for all time.

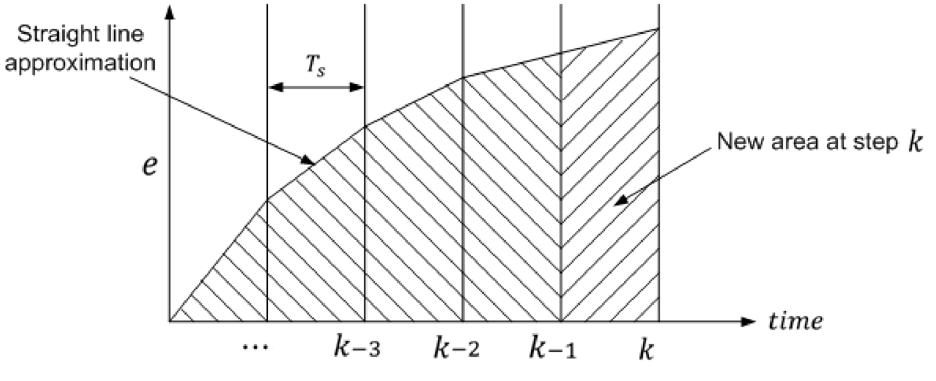


Figure 3.8 Discretisation of a continuous signal

The area is built up of incremental trapezoidal areas. The value of the integral at time step k is obtained by calculation of the latest shaded area between time step k and $k-1$ and summing (integrating) this with the previous time steps. Denoting the previous integral as P_{int} and defining the new area as $(e_k + e_{k-1})T_s/2$ leads to (for the simple Euler method) the following discretisation

$$u_k = K_p\left(e_k + T_d\frac{(e_k - e_{k-1})}{T_s} + \frac{1}{T_i}\left(\frac{(e_k + e_{k-1})}{2}T_s + P_{int}\right)\right)$$

By collecting terms involving e_k and e_{k-1} the following simplification arises

$$u_k = Ae_k + Be_{k-1} + CP_{int_k}$$

where the constants A, B and C are given by

$$A = K_p\left(1 + \frac{T_d}{T_s} + 0.5\frac{T_s}{T_i}\right)$$

$$B = K_p\left(-\frac{T_d}{T_s} + 0.5\frac{T_s}{T_i}\right)$$

$$C = \frac{K_p}{T_i}$$

Note that at each time step the value of P_{int_k} will need to be updated such that

$$P_{int_k} = P_{int_{k-1}} + 0.5(e_k + e_{k-1})T_s$$

3.4.2.1. Incremental form of a PID

There are a number of advantages of incremental control. One of these is that P_{int} is no longer required as the integral action is automatically included in the previous control value u_{k-1}. Defining $\Delta u_k = u_k - u_{k-1}$ the incremental form of the PID algorithm becomes

$$\Delta u_k = K_p\left\{e_k - e_{k-1} + T_d\frac{(e_k - 2e_{k-1} + e_{k-2})}{T_s} + \frac{1}{T_i}\frac{(e_k + e_{k-1})}{2}T_s\right\}$$

and

$$u_k = \Delta u_k + u_{k-1}$$

which is then the applied input to the plant.

3.5 Solution of the state equation

The solution of the scalar equation

$$\dot{x} = ax + bu, \qquad x(0) = x_0$$

is known to be

$$x(t) = e^{at}x_0 + \int_0^t e^{t-\tau}\, bu(\tau)dt$$

where

$$e^{at} = 1 + at + \left(\tfrac{1}{2!}\right)a^2t^2 + \left(\tfrac{1}{3!}\right)a^3t^3 + \cdots$$

By analogy for the matrix case

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{b}u$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x_0} + \int_0^t e^{\boldsymbol{A}t-\tau}\, \boldsymbol{b}u(\tau)dt$$

where the matrix exponential is defined by

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \left(\tfrac{1}{2!}\right)\boldsymbol{A}^2t^2 + \left(\tfrac{1}{3!}\right)\boldsymbol{A}^3t^3 + \cdots = \sum_{k=0}^{\infty}\frac{\boldsymbol{A}^kt^k}{k!}$$

The matrix $e^{\boldsymbol{A}t}$ is known as the state transition matrix, sometimes denoted $\boldsymbol{A}(T)$ or $\boldsymbol{\Phi}(t)$

It also relates to the resolvent matrix $(s\boldsymbol{I} - \boldsymbol{A})^{-1}$ via the inverse Laplace transform, i.e.

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\}$$

3.5.1. Properties of the state transition matrix $e^{\boldsymbol{A}t}$

$\boldsymbol{A}(t)$ has the inverse $\boldsymbol{A}^{-1}(t)$.
$e^{\boldsymbol{A}(t)}$ has the inverse $e^{-\boldsymbol{A}(t)}$

Resulting in the transitions shown in Figure 3.9.

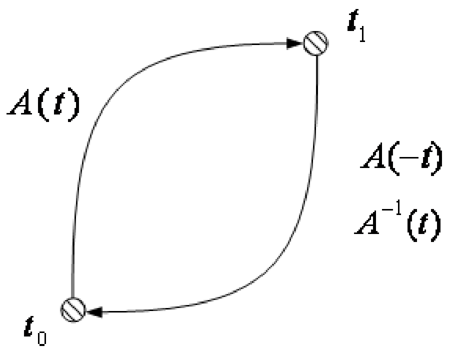


Figure 3.9 Illustrating the state transition matrix

3.5.2. Discrete time solution

When use is made of a ZOH the signal $u(t)$ becomes a 'staircase' function, constant over the intervals, where the interval is denoted T_s, and known as the sampling interval. Setting $t = T$ (dropping the subscript s for convenience)

$$\boldsymbol{x}(T) = e^{At}\boldsymbol{x}(0)|_{t=T} + \int_0^t e^{A(t-\tau)}\boldsymbol{b}u(\tau)d\tau|_{t=T}$$

3.5.3. Modal matrix approach to solution of state equation

Recall that

$$\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} = \boldsymbol{\Lambda}$$

where $\boldsymbol{M}$ is the modal matrix comprising eigenvectors as its columns and $\boldsymbol{\Lambda}$ is the diagonal spectral matrix comprising eigenvalues as the diagonal elements

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{\Lambda}$$

Application of the modal transformation in the state equation provides an additional way to calculate the state transition matrix $e^{\boldsymbol{A}t}$

Applying the transformation

$$\boldsymbol{x} = \boldsymbol{M}\boldsymbol{z} \leftrightarrow \boldsymbol{M}^{-1}\boldsymbol{x} = \boldsymbol{z}$$

$$\boldsymbol{M}\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{M}\boldsymbol{z}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}\boldsymbol{z} = \boldsymbol{\Lambda}\boldsymbol{z}$$

Since $\boldsymbol{z_0} = \boldsymbol{M}^{-1}\boldsymbol{x_0}$ the solution is

$$\boldsymbol{z}(t) = \boldsymbol{e}^{\boldsymbol{\Lambda}t}\boldsymbol{z_0}$$

$$= \boldsymbol{e}^{\boldsymbol{\Lambda}t}\boldsymbol{M}^{-1}\boldsymbol{x_0}$$

Since $\boldsymbol{z}(t) = \boldsymbol{M}^{-1}\boldsymbol{x}(t)$ it follows that

$$\boldsymbol{x}(t) = \boldsymbol{M}e^{\boldsymbol{\Lambda}t}\boldsymbol{M}^{-1}\boldsymbol{x_0}$$

Hence it immediately follows that

$$e^{At} = \boldsymbol{M}e^{\boldsymbol{\Lambda}t}\boldsymbol{M}^{-1}$$

where

$$e^{\boldsymbol{\Lambda}t} = diag(e^{\lambda_1 t}, e^{\lambda_2 t} \dots e^{\lambda_n t})$$

Note that input vector $\boldsymbol{b}(T)$ is obtained from

$$\boldsymbol{b}(T) = \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}(\tau)d\tau|_{t=T}$$

In summary, there are various methods to calculate the state transition matrix $e^{\boldsymbol{A}t}$, two methods have been demonstrated here, namely $\mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\}$ and $\boldsymbol{M}e^{\boldsymbol{\Lambda}t}\boldsymbol{M}^{-1}$. The output vector $\boldsymbol{c}$ in the output equation $y = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$ remains the same for both continuous and discrete representations.

Exercises

1. Consider the second order system represented by the differential equation:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = ku$$

Obtain the equivalent

$$s^2y + 3sy + 2y = ku$$

representation in continuous-time transfer function form

$$G(s) = \frac{u}{y} = \frac{ku}{s^2 + 3s + 2}$$

and show that the denominator may be expressed as the product of two linear factors.

$$(s + 1)(s + 2)$$

Hence deduce the poles of the system (which are also the zeros of the denominator). State which of the two poles is the dominant one.

Re-express the system in the state-space form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{b}\,u$$

$$y = \boldsymbol{c}^T\boldsymbol{x}$$

when use is made of the phase variable cannonical form.

Determine the eigenvectors corresponding to the eigenvalues and write down the modal matrix $\boldsymbol{M}$. Perform the matrix operation $\boldsymbol{M^{-1}AM}$ and comment/describe the result obtained.

Note that the states are defined by the designer/user and u and y are the system input and output, respectively. The quantities $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are defined by the choice of $\boldsymbol{x}$ and termed the system matrix, input vector and output vector, respectively.

Also note that a different realization arising from a different definition of the state variables is used within Matlab. Use the Matlab command **tf2ss** for the example and deduce the form which Matlab assumes.

2. Consider the following system:

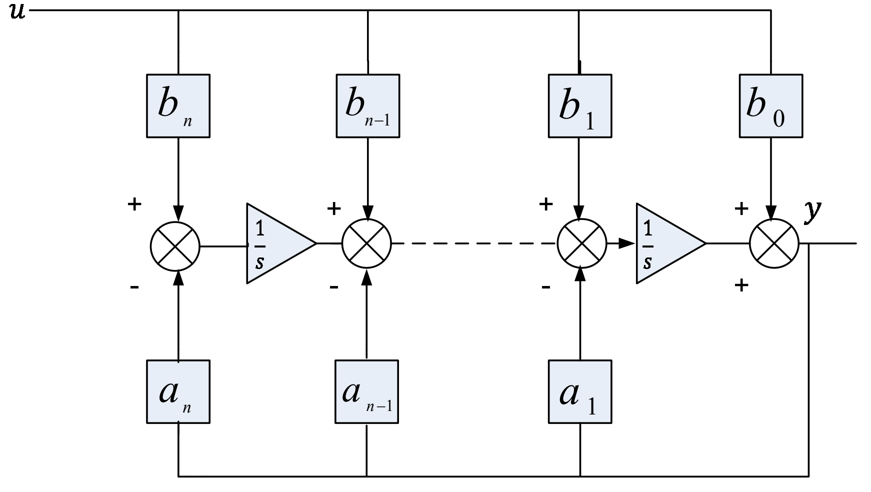
$$\ddot{y} + 3\dot{y} + 2y = k_2 u + k_1 \dot{u}$$

The system now has a zero, which is the zero of the numerator polynomial given by:

$$k_1 s + k_2 = 0$$

This will have an effect on the system dynamics as the zero will affect the system, response and will influence the closed loop characteristics. It is useful for such systems to make use of another cannonical form, termed the observable or second companion form.

Show that in general, the second companion form may be realised as follows:



General Second Companion Form

where b_0 represents a feed through term.

3. For a system having the general transfer function:

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Construct the realisation diagram in Simulink and compare the responses from the transfer function. From the definition of the state variables in the block diagram write down the resulting general form of the state space representation for this system.

Enter the state-space representation into Matlab and compare the response to the above.

4. Construct the realisation and state space representation of the following system:

$$\dddot{y} + 7\ddot{y} + 5\dot{y} + 3 = 8\dddot{u} + 6\ddot{u} + 4\dot{u} + 2u$$

Comment on the form of the output equation for this system. Note also that the transfer function $G(s)$ is unrealizable in practice.

5. Consider a different representation (introducing slightly different notation).

$$G(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

show that upon dividing through that this will give

$$G(s) = \frac{c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \cdots + c_1 s + c_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} + b_n$$

deduce that

$$b_{n-1} = a_{n-1} b_n + c_{n-1}$$

$$b_{n-2} = a_{n-2} b_n + c_{n-2}$$

$$\vdots$$

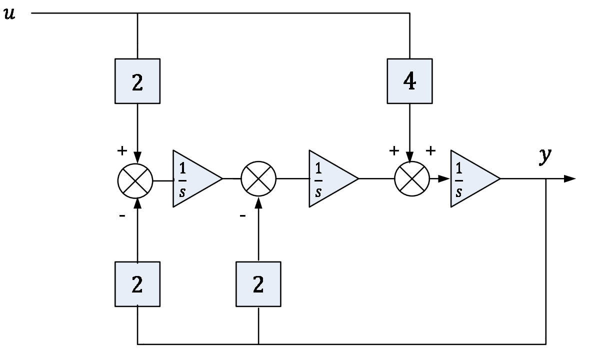
$$b_0 = a_0 b_n + c_0$$

Re-express in phase variable canonical form and show that the output equation can be expressed as:

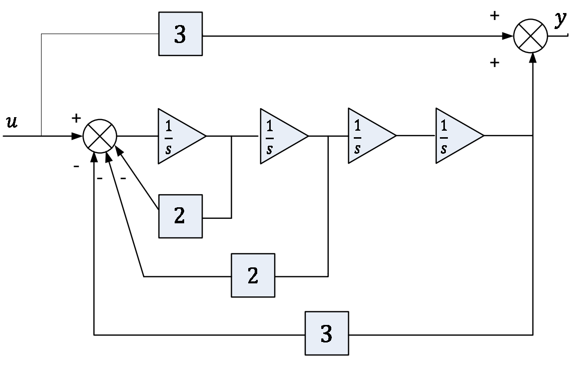
$$y = [c_0 \quad c_1 \quad \cdots \quad c_{n-1}]\, \boldsymbol{x} + b_n u$$

6. Write down the state space representation in the observable state space form; obtain the transfer function and differential equation for the following.

a)



b)



7. Determine if the following are state controllable:

a) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$

b) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$

c) $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

d) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

8. Consider the state-space model representation. Determine whether this system representation is state controllable

$\boldsymbol{A} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\boldsymbol{C} = [1 \quad 2]$, $\boldsymbol{D} = [0]$

In this form it is not completely state controllable. This will now be shown in the following example.

$$\boldsymbol{K}_C = [\boldsymbol{B} \;\vdots\; \boldsymbol{A}\,\boldsymbol{B}] = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}$$

Where the $\boldsymbol{K}_C$ matrix clearly has $rank\ (\boldsymbol{K}_C) = 1$.

Depending on whether the representation is controllable or not, determine the transfer function $G(s)$ via

$$G(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Hence or otherwise show that the system may be represented as

$$G(s) = \frac{6s + 4}{s^2 + 3s + 2}$$

Hence or otherwise deduce that the system is completely controllable in this form.

Note that state controllability of a SISO system is not dependent on the system, but rather its representation.

9. Determine which of the following are completely state observable and/or controllable

a) $\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{C} = [2 \quad 0]$,

b) $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

c) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{C} = [1 \quad 0 \quad 1]$

d) $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{C} = [0 \quad 0 \quad 1]$

10. The reslovent matrix can also be expressed as:

$$(\lambda\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{\mathrm{adj}(\lambda\boldsymbol{I} - \boldsymbol{A})}{|\lambda\boldsymbol{I} - \boldsymbol{A}|}$$

Check that the adjoint matrix $adj(\lambda\boldsymbol{I} - \boldsymbol{A})$ is a matrix polynomial of the form:

$$adj(\lambda\boldsymbol{I} - \boldsymbol{A}) \ = \boldsymbol{E}_1\lambda^{n-1} + \boldsymbol{E}_2\lambda^{n-2} + \cdots + \boldsymbol{E}_n$$

where $\boldsymbol{E}_i, i = 1 \dots n$, are $n \times n$ matrices.

Rearrange the formula by multiplying by $|\lambda\boldsymbol{I} - \boldsymbol{A}|$:

$$(\lambda\boldsymbol{I} - \boldsymbol{A})^{-1}|\lambda\boldsymbol{I} - \boldsymbol{A}| = \sum_{i=1}^{n} \boldsymbol{E}_i\lambda^{n-i}$$

Multiplying both sides by $(\lambda\boldsymbol{I} - \boldsymbol{A})$ gives

$$|\lambda\boldsymbol{I} - \boldsymbol{A}|\boldsymbol{I} = (\lambda\boldsymbol{I} - \boldsymbol{A})\sum_{i=1}^{n} \boldsymbol{E}_i\lambda^{n-i}$$

a) Show that the left hand side becomes:

$$\lambda^n\boldsymbol{I} + a_1\lambda^{n-1}\boldsymbol{I} + \cdots + a_n\boldsymbol{I}$$

b) Show that the right hand side becomes:

$$\sum_{i=2}^{n}(\boldsymbol{E}_i\lambda^{n-i} - \boldsymbol{A}\,\boldsymbol{E}_{i-1}) + \boldsymbol{E}_1\lambda^n - \boldsymbol{A}\,\boldsymbol{E}_n$$

Hence or otherwise, by equating coefficients of like powers of λ show that:

$$\begin{gathered} \boldsymbol{E}_1 = \boldsymbol{I} \\ \boldsymbol{E}_2 - \boldsymbol{A}\,\boldsymbol{E}_1 = a_1\boldsymbol{I} \\ \boldsymbol{E}_3 - \boldsymbol{A}\,\boldsymbol{E}_2 = a_2\boldsymbol{I} \\ \vdots \\ \boldsymbol{E}_n - \boldsymbol{A}\,\boldsymbol{E}_{n-1} = a_{n-1}\boldsymbol{I} \\ -\boldsymbol{A}\,\boldsymbol{E}_n = a_n\boldsymbol{I} \end{gathered}$$

This relationship leads to an important theorem of matrix algebra known as the Cayley-Hamilton theorem.

Noting that $\boldsymbol{E}_1 = \boldsymbol{I}$ it follows that:

$$\boldsymbol{E}_2 = \boldsymbol{A}\,\boldsymbol{E}_1 + a_1\boldsymbol{I} = \boldsymbol{A}\,\boldsymbol{I} + a_1\boldsymbol{I} = \boldsymbol{A} + a_1\boldsymbol{I}$$

$$\boldsymbol{E}_3 = \boldsymbol{A}\,\boldsymbol{E}_2 + a_2\boldsymbol{I} = \boldsymbol{A}(\boldsymbol{A} + a_1\boldsymbol{I}) + a_2\boldsymbol{I}$$

$$= \boldsymbol{A}^2 + a_1\boldsymbol{A} + a_2\boldsymbol{I}$$

11. From these results show that:

$$\boldsymbol{E}_4 = \boldsymbol{A}\,\boldsymbol{E}_3 + a_3\boldsymbol{I}$$

$$= \boldsymbol{A}^3 + a_1\boldsymbol{A}^2 + a_2\boldsymbol{A} + a_3\boldsymbol{I}$$

The general formula is:

$$\boldsymbol{E}_{\,n} = \boldsymbol{A}^{n-1} + a_1\boldsymbol{A}^{n-2} + \cdots + a_{n-1}\boldsymbol{I}$$

When the general formula is multiplied by matrix $\underline{A}$:

$$\boldsymbol{A}\,\boldsymbol{E}_{\,n} = \boldsymbol{A}^{n} + a_1\boldsymbol{A}^{n-1} + \cdots + a_{n-1}\boldsymbol{A}$$

With the observation that $\boldsymbol{A}\,\boldsymbol{E}_n + a_n\boldsymbol{I} = \boldsymbol{0}$ it follows that by adding $a_n\boldsymbol{I}$ the resulting formula is:

$$\boldsymbol{A}^{n} + a_1\boldsymbol{A}^{n-1} + \cdots + a_{n-1}\boldsymbol{A} + a_n\boldsymbol{I} = \boldsymbol{0}$$

This equation is the same as the characteristic equation with the scalar λ^i replaced by the matrix $\boldsymbol{A}^i$.

12. Similar results from the previous section can be applied to discrete system representations. One particularly interesting/important result is that of obtaining the discrete state-space models using the Cayley-Hamilton theorem. Consider the discrete state space sytem

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(t)\boldsymbol{x}_k + \boldsymbol{b}(t)u_k$$

$$y_{k+1} = \boldsymbol{c}^T\boldsymbol{x}_k$$

where $\boldsymbol{A}(t)$ and $\boldsymbol{b}(t)$ are functions of the sampling interval T. $\boldsymbol{A}(t)$ is the state transition matrix. $t = T_S$

$$\boldsymbol{A}(t) = e^{\boldsymbol{A}t}\Big|_{t=T_S}$$

$$\boldsymbol{b}(t) = \left.\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}d\tau\right|_{t=T_S}$$

Note that $\boldsymbol{c}$ is the same in both continuous and discrete time.

It has been found for a particular continuous-time system that using the resolvent matrix, the modal matrix as well as a method using the Cayley Hamilton theorem that

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & -0.7 \\ 1 & 1.5 \end{bmatrix}$$

i.e. the characteristic equation:

$$z^2 - 1.5z + 0.7 = 0$$

or equivalently

$$\lambda^2 - 1.5\lambda + 0.7 = 0$$

Replace λ by $\boldsymbol{A}(t)$ and find matrix $\boldsymbol{A}^{-1}(t)$. Check your results.

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Acknowledgements

The authors gratefully acknowledge the contributions of Jonathan Herring, Pawel Kret and Ivan Zajic.