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Janos Polonyi

CLASSICAL FIELD THEORY

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Chapter 1

Introduction

The following is a short notes of lectures about classical field theory, in particular classical electrodynamics for fourth or fifth year physics students. It is not supposed to be an introductory course to electrodynamics whose knowledge will be assumed. Our main interest is the consider electrodynamics as a particular, relativistic field theory. A slightly more detailed view of back reaction force acting on point charges is given, being the last open chapter of classical electrodynamics.

The concept of classical field emerged in the nineteenth century when the proper degrees of freedom have been identified for the electromagnetic interaction and the idea was generalized later. A half century later the careful study of the propagation of the electromagnetic waves led to special relativity. One is usually confronted with relativistic effects at high energies as far as massive particles are concerned and the simpler, non-relativistic approximation is sufficient to describe low energy phenomena. But a massless particle, such as the photon, moves with relativistic speed at arbitrarily low energy and requires the full complexity of the relativistic description.

We do not follow here the historical evolution, rather start with a very short summary of the main idea of special relativity. This makes the introduction of classical field more natural. Classical field theories will be introduced by means of the action principle. This is not only a rather powerful scheme but it offers a clear view of the role symmetries play in the dynamics. After having laid down the general formalism we turn to the electrodynamics, the interactive system of point charges and the electromagnetic field. The presentation is closed by a short review of the state of the radiation back reaction force acting on accelerating point charges.

This lecture notes differs from a text book to be written about classical field theory in restricting the attention to subjects which can be covered in a one semester course and as a result gauge theory in general and in particular general relativity are not presented. Another difference is the inclusion of a subject, special relativity, which might not be presented in other courses.

There are numerous textbooks available in this classical subject. The monograph [1] is monumental collection of different aspects of electrodynamics, the basics can be found best in [2]. The radiation reaction force is nicely discussed in [3], and [4].

Chapter 2

Elements of special relativity

The main concepts of special relativity are introduced in this chapter. They caused a genuine surprise a century ago because people had the illusion that their intuition, based on the physics of slow moving object, covers the whole range of Physics.

The deviation from Newton's mechanics of massive bodies has systematically been established few decades after the discovery of special relativity only. In the meantime the only strong evidence of special relativity came from electromagnetic radiation, from the propagation of massless particles, the photons. They move with the speed of light at any energy and provide ample evidences of the new physics of particles moving with speed comparable with the speed of light. Therefore we rely on the propagation of light signals in the discussions below without entering into the more detailed description of such signals by classical electrodynamics, the only reference to the Maxwell equations being made in the simple assumption 2 below.

2.1 Newton's relativity

A frequently used concept below is the inertial coordinate systems. Simplest motion is that of a free particle and the inertial coordinate systems are where a free point particle moves with constant velocity. Once the motion of a free particle satisfy the same equation, vanishing acceleration, in each inertial systems one conjectures that any other, interactive system follow the same laws in different inertial systems. Newton's law, $m\ddot{\mathbf{x}} = -\nabla U$, includes the second time derivative of the coordinates, therefore inertial systems are connected by motion of constant speed,

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} - t\mathbf{v}. \tag{2.1}$$

This transformation is called Galilean boost because the invariance of the laws of mechanics under such transformation, the relativity assumption of Newton's theory, was discovered by Galileo. In other words, there is no way to find out the absolute velocity in mechanics because the physical phenomena found by two observers, moving with constant velocity with respect to each other are identical.

The point which marks the end of the applicability of Newton's theory in physics is which was assumed for hundreds of years but left implicit in Galilean boost, namely that the time remains the same,

$$t \to t' = t \tag{2.2}$$

when an inertial system is changed into another one. In other words, the time is absolute in Newton's physics, can in principle be introduced for all inertial system identically.

2.2 Conflict resolution

Special relativity results from the solution of a contradiction among the two main pillars of classical physics, mechanics and electrodynamics.

The following two assumptions seem to be unacceptable:

1. Principle of Newton's relativity: The laws of Physics look the same in the inertial coordinate systems.

2. Electrodynamics: According to the Maxwell equations the speed of the propagation of electromagnetic waves (speed of light) is $c = 2.99793 \cdot 10^{10}$cm/s.

In fact, the Galilean boost of Eqs. (2.1)-(2.2) leads to the addition of velocities, $\frac{d\mathbf{x}'}{dt} = \frac{d\mathbf{x}}{dt} - \mathbf{v}$. This result is in contradiction with the inertial system independence of the speed of light, encoded in the Maxwell-equations.

It is Einstein's deep understanding physics which led him to recognize that Eq. (2.2) is the weak point of the argument, not supported by observations and special relativity is based on its rejection. Special relativity is based on the following, weakened assumptions.

1' There is a transformation $\mathbf{x} \to \mathbf{x}'$ and $t \to t'$ of the coordinate and time which maps an inertial system into another and preserves the laws of physics. This transformation changes the observed velocity of objects, rendering impossible to measure absolute velocities.

2' The speed of light is the same in every intertial system.

Once the time lost its absolute nature then the next step is its construction for each inertial system by observations. After this point is completed one can clarify the details of the relation mentioned in assumption 1', between the time and coordinates when different inertial systems are compared. This will be our main task in the remaining part of this chapter.

The loss of absolute nature of the time forces us to change the way we imagine the motion of an object. In the Newtonian mechanics the motion of a point particle was characterized by its trajectory $\mathbf{x}(t)$, its coordinates as the function of the (absolute) time. If the time is to be constructed in a dynamical manner then one should be more careful and not use the same time for different objects. Therefore, the motion of a point particle is described by its world line $x^\mu = (ct(s), \mathbf{x}(s))$, $\mu = 0, 1, 2, 3$, the parametrized form of its time and coordinates. The trivial factor c, the speed of light, is introduced for the time to have components with the same length dimensions in the four-coordinate $x^\mu(s)$. Each four-coordinate labels a point in the space-time, called event. The world line of a point particle is a curve in the space time.

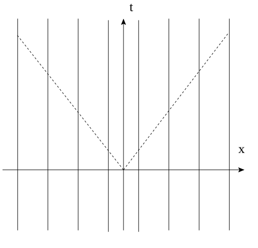

Figure 2.1: Synchronization of clocks to the one placed at the origin.

Let us suppose that we can introduce a coordinate system by means of meter rods which characterize points in space and all are in rest. Then we place a clock at each space point which will be synchronized in the following manner. We pick the clock at one point, $\mathbf{x} = 0$ in Fig. 2.1, as a reference, its finger being used to construct the flow of time at $\mathbf{x} = 0$, the time variable of its world line. Suppose that we want now to set the clock at point $\mathbf{y}$. We first place a mirror on this clock and then emit a light signal which propagates with the speed of light according to assumption 2' from our reference point at time t_0 and measure the time t_1 when it arrives back from $\mathbf{y}$. The clock at $\mathbf{y}$ should show the time $(t_1 - t_0)/2$ when the light has just reached.

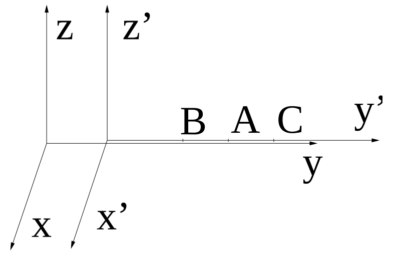

Figure 2.2: The arrival of the light to B and C are simultaneous ($|AB|' = |AC|'$) in the inertial system (ct, x, y, x) but the light signals arrive earlier to B than C in the inertial system (ct', x', y', x').

The clocks, synchronized in such a manner show immediately one of the most dramatic prediction of special relativity, the loss of absolute nature of time. Let us imagine an experimental rearrangement in the coordinate system (x, y, z) of Fig. 2.2 which contains a light source (A) and two light detectors (B and C), placed at equal distance from the source. A light signal, emitted form the source reaches the detectors at the same time in this intertial system. Let us analyze the same process seen from another inertial system (ct', x', y', z') which is attached to an observer moving with a constant velocity in the direction of the y axis. A shift by a constant velocity leaves the free particle motion unaccelerated therefore the coordinate system (ct', x', y', z') where this observes is at rest is inertial, too. But the time ct' when the detector C signals the arrival of the light for this moving observer is later than the time ct in the co-moving inertial system. In fact, the light propagates with the same speed in both systems but the detector moves away form the source int the system (ct', x', y', z'). In a similar manner, the time ct' when the light reaches detector B is earlier than ct because this detector moves towards the source. As a result, two events which are in coincidence in one inertial system may correspond to different times in another inertial system. The order of events may change when we see them in different inertial systems where the physical laws are supposed to be identical.

2.3 Invariant length

The finding of the transformation rule for space-time vectors $x^\mu = (ct, \mathbf{x})$ is rendered simpler by the introduction of some kind of length between events which is the same when seen form different inertial systems. Since the speed of light is the same in every inertial system it is natural to use light in the construction of this length. We define the distance between two events in such a manner that is is vanishing when there is a light signal which connects the two events. The distance square is supposed to be quadratic in the difference of the space-time coordinates, thus the expression

$$s^2 = c^2(t_2 - t_1)^2 - (\mathbf{x}_2 - \mathbf{x}_1)^2. \tag{2.3}$$

is a natural choice. If s^2 is vanishing in one reference frame then the two events can be connected by a light signal. This property is valid in any reference frame, therefore the value $s^2 = 0$ remains invariant during change of inertial systems.

Now we show that $s^2 \neq 0$ remains invariant, as well. The change of inertial system may consist of trivial translations in space-time and spatial rotation which leave the the expression (2.3) unchanged in an obvious manner. What is left to show is that a relativistic boost of the inertial system when it moves with a constant speed leaves $s^2 \neq 0$ invariant.

The so far unspecified transformation of the space-time coordinates between two intertial systems related by a relativistic boost of velocity $\mathbf{v}$ is supposed to generate a transformation $s^2 \to s'^2 = F(s^2, \mathbf{u})$ during the boost. We know that $F(0, \mathbf{v}) = 0$ and assume that the transformation rule is free of singularities and the function $F(s^2, \mathbf{v})$ has a Taylor expansion around $s^2 = 0$,

$$F(s^2, \mathbf{v}) = a(|\mathbf{v}|)s^2 + \mathcal{O}\left(s^4\right). \tag{2.4}$$

where

$$a(|\mathbf{v}|) = \frac{\partial F(0, \mathbf{v})}{\partial s^2}. \tag{2.5}$$

Note that the symmetry under spatial rotations requires that the three-dimensional scalar $a(|\mathbf{v}|)$ depends on the length of the three-vector $\mathbf{v}$ only.

Let us no consider three reference frames S, $S(\mathbf{u}_1)$ and $S(\mathbf{u}_2)$ the two latter moving with infinitesimal velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ with respect to S. Because of $s^2 = 0$ is invariant and the transformation law for s^2 should be continuous in $\mathbf{u}$ for infinitesimal ds^2 (no large distances involved where physical phenomena might accumulate) we have

$$\begin{aligned} ds^2 &= a(|\mathbf{u}_1|)ds_1^2, \\ ds^2 &= a(|\mathbf{u}_2|)ds_2^2, \end{aligned} \tag{2.6}$$

where $a(u)$ is a continuous function and the argument depends on the magnitude $|\mathbf{u}|$ only owing to rotational invariance. When $S(\mathbf{u}_1)$ is viewed from

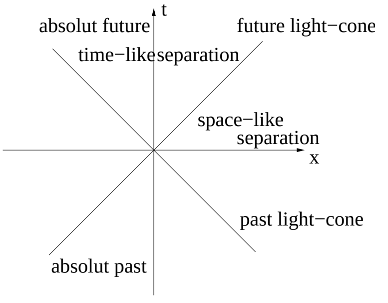

Figure 2.3: The light cones.

$S(\mathbf{u}_1)$ then one finds

$$ds_1^2 = a(|u_1 - u_2|)ds_2^2 \tag{2.7}$$

and the comparison of (2.6) and (2.7) gives

$$a(|\mathbf{u}_1 - \mathbf{u}_2|) = \frac{a(|\mathbf{u}_2|)}{a(|\mathbf{u}_1|)} \tag{2.8}$$

which can be true only if $a = 1$. This argument, repeated for successively applied Lorentz-boosts establishes the invariance of the length square such changes of reference system which can be reached by repeated infinitesimal transformations.

One says that two events are time-, space- or light-like separated when $s^2 > 0$, $s^2 < 0$ or $s^2 = 0$, respectively. Signals emitted from a point, shown as the origin in Fig. 2.3 reaches the future light cone. The signals received may be emitted form its past light cone. There is no communication between two events when they are space-like. Events separated by light-like interval can communicate by signals traveling with the speed of light only.

2.4 Lorentz Transformations

The use of the invariant length is a simple characterization of the transformation of the space-time coordinates when the inertial system is changed, a Lorentz transformation is carried out. For this end we introduce the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.9}$$

which allows us to introduce a Lorentz-invariant scalar product

$$x \cdot y = x^\mu g_{\mu\nu} y^\nu \tag{2.10}$$

where $x = (ct, \mathbf{x})$, etc. The Lorentz-group consists of 4×4 matrices which mix the space-time coordinates

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{2.11}$$

in such a manner that the scalar product or the invariant length is preserved,

$$x \cdot y = x^{\mu'} \Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'} y^{\nu'} \tag{2.12}$$

or

$$g = \tilde{\Lambda} \cdot g \cdot \Lambda. \tag{2.13}$$

The Lorentz group is 6 dimensional, 3 dimensions correspond to three-dimensional rotations and three other directions belong to Lorentz-boosts, parametrized by the three-velocity $\mathbf{v}$ relating the inertial systems. let us denote the the parallel and perpendicular projection of the three-coordinate on the velocity $\mathbf{v}$ by $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$, respectively,

$$\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp, \quad \mathbf{x}_\parallel \cdot \mathbf{x}_\perp = \mathbf{v} \cdot \mathbf{x}_\perp = 0. \tag{2.14}$$

We can then write a general Lorentz transformation in a three-dimensional notation as

$$\mathbf{x}' = \alpha(\mathbf{x}_\parallel - \mathbf{v}t) + \gamma \mathbf{x}_\perp, \qquad t' = \beta\left(t - \frac{\mathbf{x} \cdot \mathbf{v}}{\tilde{c}^2}\right) \tag{2.15}$$

The invariance of the length,

$$c^2 t^2 - \mathbf{x}^2 = c^2 \beta^2 \left(t - \frac{\mathbf{x} \cdot \mathbf{v}}{\tilde{c}^2}\right)^2 - \alpha^2 (\mathbf{x}_\parallel - \mathbf{v}t)^2 - \gamma \mathbf{x}_\perp^2, \tag{2.16}$$

yields the relations

$$\begin{aligned} \gamma &= \pm 1, \quad v = 0 \implies \gamma = 1 \\ \tilde{c} &= c \\ \alpha &= \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned} \tag{2.17}$$

$$x'_\parallel = \frac{x_\parallel - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{v x_\parallel}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.18}$$

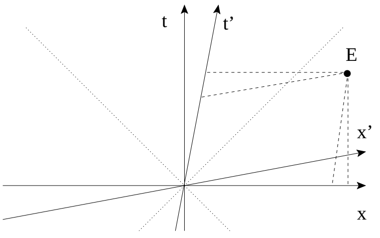

Figure 2.4: Lorentz transformations.

Note that the inverse Lorentz transformation is obtained by the change $v \to -v$,

$$x_\parallel = \frac{x'_\parallel + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{t' + \frac{vx'_\parallel}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.19}$$

Fig. 2.4 shows that change of the space-time coordinates during a Lorentz boost. For an Euclidean rotation in two dimensions both axes are rotated by the same angle, here this possibility is excluded by the invariance of the light cone. As a results the axes are moved by keeping the light cone, shown with dashed lines, unchanged.

We remark that there are four disconnected components of the Lorentz group. First note that the determinant of Eq. (2.13), $\det g = \det g (\det \Lambda)^2$ indicates that $\det \Lambda = \pm 1$ and there are no infinitesimal Lorentz transformations $\mathbb{1} + \delta\Lambda$ such that $\det \Lambda(\mathbb{1} + \delta\Lambda) \neq \det \Lambda$. Thus the spatial inversion split the Lorentz group into two disconnected sets. Furthermore, observe that the component (00) of Eq. (2.13), $1 = g_{00} = (\Lambda^0{}_0)^2 - \sum_j (\Lambda^j{}_0)^2$ implies that $\Lambda^0{}_0| > 1$, and that time inversion, a Lorentz transformation, splits the :Lorentz group into two disconnected sets. The four disconnected components consists of matrices satisfying Eq. (2.13) and

1. $\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$ (the proper Lorentz group, $L^\uparrow_+$),

2. $\det \Lambda = 1$, $\Lambda^0{}_0 \leq 1$,

3. $\det \Lambda = -1$, $\Lambda^0{}_0 \geq 1$,

4. $\det \Lambda = -1$, $\Lambda^0{}_0 \leq 1$.

Note that one recovers the Galilean boost, $x' = x - vt$, in the non-relativistic limit. The argument for the invariance of the length s^2, presented in Chapter

2.3 applies for $L_+^\uparrow$ only. But inversions preserve s^2 in a obvious manner therefore, the invariance holds for the whole Lorentz group.

One usually needs the full space-time symmetry group, called Poincar group. It is ten dimensional and is the direct product of the six dimensional Lorentz group and the four dimensional translation group in the space-time.

2.5 Time dilatation

The proper time τ is the lapse the time measured the coordinate system attached to the system. To find it for an object moving with a velocity $\mathbf{v}$ to be considered constant during a short motion, in a reference system let us express the invariant length between two consecutive events,

$$\text{ref. system of the particle} \quad c^2 d\tau^2 = c^2 dt^2 - dt^2 \mathbf{v}^2 \quad \text{lab. system} \tag{2.20}$$

which gives

$$d\tau = dt\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}. \tag{2.21}$$

Remarks:

1. A moving clock seems to be slower than a standing one.
2. The time measured by a clock,

$$\frac{1}{c}\int_{x_i}^{x_f} ds \tag{2.22}$$

   is maximal if the clock moves with constant velocity, ie. its world-line is straight. (Clock following a motion with the same initial and final point but non-constant velocity seems to be slower than the one in uniform motion.)

2.6 Contraction of length

The proper length of a rod, $\ell_0 = x'_2 - x'_1$, is defined in the inertial system S' in which the rod is at rest. In another inertial system the end points correspond to the world lines

$$x_j = \frac{x'_j + vt'_j}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t_j = \frac{t'_j + \frac{vx'_j}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.23}$$

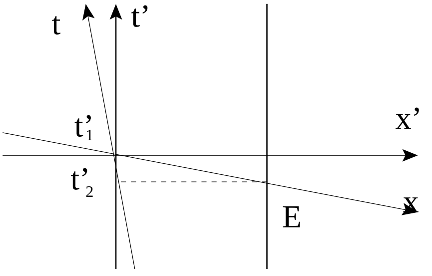

Figure 2.5: Lorentz contraction.

The length is read off at equal time, $t_1 = t_2$, thus

$$t'_2 - t'_1 = -\frac{v}{c^2}(x'_2 - x'_1) = -\frac{v\ell_0}{c^2} \tag{2.24}$$

and the invariant length of the space-time vector pointing to the event E is

$$-\ell^2 = c^2\left(\frac{v\ell_0}{c^2}\right)^2 - \ell_0^2, \tag{2.25}$$

yielding

$$\ell = \ell_0\sqrt{1 - \frac{v^2}{c^2}}. \tag{2.26}$$

Lorentz contraction is that the length is the longest in the rest frame. It was introduced by Lorentz as an ad hoc mechanism to explain the negative result of the Michelson-Moreley experiment to measure the absolute speed of their laboratory. It is Einstein's essential contribution to change this view and instead of postulating a fundamental effect he derived it by the detailed analysis of the way length are measured in moving inertial system. Thus the contraction of the length has nothing to do with real change in the system, it reflects the specific features of the way observations are done only.

2.7 Transformation of the velocity

As mentioned above, the Galilean boost (2.1)-(2.2) leads immediately to the addition of velocities, $\frac{d\mathbf{x}}{dt} \to \frac{d\mathbf{x}}{dt} - \mathbf{v}$. This rule is in contradiction with the invariance of the speed of light under Lorentz boosts. It was mentioned that the resolution of this conflict is the renounce of the absolute nature of the time. This must introduce non-linear pieces in the transformation law of the velocities. To find them we denote by V the velocity between the inertial systems S and S',

$$dx_\parallel = \frac{dx'_\parallel + V dt'}{\sqrt{1-\frac{V^2}{c^2}}}, \quad dx_\perp = dx'_\perp, \quad dt = \frac{dt' + \frac{V dx'_\parallel}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}. \tag{2.27}$$

Then

$$\frac{dt}{dt'} = \frac{1+\frac{Vv'_\parallel}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \tag{2.28}$$

and the velocity transform as

$$v_\parallel = \frac{v'_\parallel + V}{1+\frac{Vv'_\parallel}{c^2}}, \quad v_\perp = v'_\perp \frac{\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{Vv'_\parallel}{c^2}}. \tag{2.29}$$

Note that

1. the rule of addition of velocity is valid for $v/c \ll 1$,
2. if $v = c$ then $v' = c$,
3. the expressions are not symmetrical for the exchange of v and V

2.8 Four-vectors

The space-time coordinates represent the contravariant vectors $x^\mu = (ct, \mathbf{x})$. In order to eliminate the metric tensor from covariant expressions we introduce covariant vectors whose lower index is obtained by multiplying with the metric tensor, $x_\mu = g_{\mu\nu} x^\nu$. Thus allows us to leave out the metric tensor from the scalar product, $x \cdot y = x^\mu g_{\mu\nu} y^\nu = x^\mu y_\mu$. The inverse of the metric tensor $g_{\mu\nu}$ is denoted by $g^{\mu\nu}$, $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$.

Identities for Lorentz transformations:

$$\begin{aligned}
g &= \tilde{\Lambda} \cdot g \cdot \Lambda \\
\Lambda^{-1} &= g^{-1} \cdot \tilde{\Lambda} \cdot g = (g \cdot \Lambda \cdot g^{-1})^{\mathrm{tr}} \\
x'^\mu &= (\Lambda \cdot x)^\mu = \Lambda^\mu{}_\nu x^\nu \\
x^\mu &= (g \cdot \Lambda \cdot g^{-1})_\nu{}^\mu x'^\nu = x'^\nu \Lambda_\nu{}^\mu = (x' \cdot \Lambda)^\mu \\
x'_\mu &= (g \cdot \Lambda \cdot x)^\mu = (g \cdot \Lambda \cdot g^{-1} \cdot g \cdot x)^\mu = \Lambda_\mu{}^\nu x_\nu \\
x_\mu &= x'_\nu \Lambda^\nu{}_\mu = (x' \cdot \Lambda)_\mu
\end{aligned} \tag{2.30}$$

One can define contravariant tensors which transform as

$$T^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}T^{\nu_1\cdots\nu_n}, \tag{2.31}$$

covariant tensors with the transformation rule

$$T_{\mu_1\cdots\mu_n} = \Lambda^{\nu_1}{}_{\mu_1}\cdots\Lambda^{\nu_n}{}_{\mu_n}T_{\nu_1\cdots\nu_n} \tag{2.32}$$

and mixed tensors which satisfy

$$T^{\rho_1\cdots\rho_m}_{\mu_1\cdots\mu_n} = \Lambda^{\rho_1}{}_{\kappa_1}\cdots\Lambda^{\rho_m}{}_{\kappa_m}\Lambda^{\nu_1}{}_{\mu_1}\cdots\Lambda^{\nu_n}{}_{\mu_n}T^{\kappa_1\cdots\kappa_m}_{\nu_1\cdots\nu_n}. \tag{2.33}$$

There are important invariant tensors, for instance the metric tensor is preserved, $g_{\mu\nu'} = \Lambda^{\mu'}{}_{\mu}g_{\mu'\nu'}\Lambda^{\nu'}{}_{\nu}$ together with its other forms like $g_{\mu\nu}$, $g^{\mu\nu}$ and g^{ν}_{μ}. Another important invariant tensor is the completely antisymmetric one $\epsilon^{\mu\nu\rho\sigma}$ where the convention is $\epsilon^{0123} = 1$. In fact, $\epsilon^{\mu\nu\rho\sigma'} = \epsilon^{\mu\nu\rho\sigma}\det\Lambda$ which shows that $\epsilon^{\mu\nu\rho\sigma}$ is a pseudo tensor, is remains invariant under proper Lorentz transformation and changes sign during inversions.

2.9 Relativistic mechanics

Let us first find the heuristic generalization of Newton's law for relativistic velocities by imposing Lorentz invariance. The four-velocity is defined as

$$u^\mu = \frac{dx^\mu(s)}{ds} = \dot{x}(s) = \left(\frac{dx^0}{ds}, \frac{dx^0}{ds}\frac{\mathbf{v}}{c}\right) = \left(\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}, \frac{\frac{\mathbf{v}}{c}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right) \tag{2.34}$$

and it gives rise the four-acceleration

$$\dot{u}^\mu = \frac{du^\mu}{ds}, \tag{2.35}$$

and the derivation of the identity $u^2(s) = 1$ with respect to s yields $\dot{u}\cdot u = 0$.

The four-momentum, defined by

$$p^\mu = mcu^\mu = (p^0, \mathbf{p}) = \left(\frac{mc}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}, \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right), \tag{2.36}$$

satisfies the relation $p^2 = m^2c^2$. The rate of change of the four-momentum defines the four-force,

$$K^\mu = \frac{dp^\mu}{ds} = \frac{d}{ds}\left(mc\frac{dx^\mu}{ds}\right). \tag{2.37}$$

The three-vector

$$
\begin{aligned}
\mathbf{F} &= \frac{ds}{dt}\mathbf{K} \\
&= mc\frac{d}{dt}\frac{dt}{ds}\mathbf{v} \\
&= \frac{m\mathbf{a}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} - \frac{\frac{d^2s}{dt^2}}{(\frac{ds}{dt})^2}mc\mathbf{v} \\
&= \frac{m\mathbf{a}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} - \frac{\frac{d}{dt}\sqrt{c^2-\mathbf{v}^2}}{c^2-\mathbf{v}^2}mc\mathbf{v} \\
&= \frac{m}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\left[\mathbf{a} + \frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{a})}{c^2(1-\frac{\mathbf{v}^2}{c^2})}\right] \qquad (2.38)
\end{aligned}
$$

can be considered as the relativistic generalization of the the three-force in Newton's equation. The particular choice of $\mathcal{O}\left(\mathbf{v}^2/c^2\right)$ corrections are chosen in such manner that the temporal component of Eq. (2.37),

$$
\frac{d}{ds}\left(mc\frac{dx^0}{ds}\right) = \frac{d}{ds}\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = K^0 \qquad (2.39)
$$

leads to the conservation law for the energy. This is because the constraint $0 = mc\dot{u}\cdot u = K\cdot u = mc\ddot{x}\cdot\dot{x} = 0$ gives

$$
K^0\frac{dx^0}{ds} = \mathbf{Ku} = \left(\frac{dt}{ds}\right)^2\mathbf{Fv} \qquad (2.40)
$$

what can be written as

$$
\frac{d}{dt}E(\mathbf{v}) = \mathbf{Fv} \qquad (2.41)
$$

which gives the kinetic energy

$$
E(\mathbf{v}) = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \qquad (2.42)
$$

and leads to the expressions

$$
p^\mu = \left(\frac{E}{c}, \mathbf{p}\right), \qquad \frac{E^2}{c^2} = \mathbf{p}^2 + m^2c^2, \qquad E(\mathbf{p}) = c\sqrt{\mathbf{p}^2+m^2c^2}. \qquad (2.43)
$$

Note that the unusual relativistic correction in the three-force (2.38) is non-vanishing when the velocity is not perpendicular to the acceleration, i.e. the kinetic energy is not conserved and work done by the force on the particle.

2.10 Lessons of special relativity

Special relativity grew out from the unsuccessful experimental attempts of measuring absolute velocities. This negative results is incorporated into the dynamics by postulating a symmetry of the fundamental laws in agreement with Maxwell equations. The most radical consequences of this symmetry concerns the time. It becomes non-absolute, has to be determined dynamically for each system instead of assumed to be available before any observation. Furthermore, two events which coincide in one reference frame may appear in different order in time in other reference frames, the order of events in time is not absolute either. The impossibility of measuring absolute acceleration and further, higher derivatives of the coordinates with respect to the time is extended in general relativity to the nonavailability of the coordinate system before measurements where the space-time coordinates are constructed by the observers.

The dynamical origin of time motivates the change of the trajectory $\mathbf{x}(t)$ as a fundamental object of non-relativistic mechanics to world line $x^\mu(s)$ where the reference system time x^0 is parametrized by the proper time or simply a parameter of the motion s. The world line offers a surprising extension of the non-relativistic motion by letting $x^0(s)$ non-monotonous function. Turning point where time turns back along the world line is interpreted in the quantum case as an events where a particle-anti particle pair is created or annihilated.

We close this short overview of special relativity with a warning. The basic issues of this theory , such as meter rods and clocks are introduced on the macroscopic level. Though the formal implementation of special relativity is fully confirmed in the quantum regime their interpretation in physical term, e.g. the speed of propagation of light within an atom, is neither trivial nor parallel with the macroscopic reasoning.

Chapter 3

Classical Field Theory

3.1 Why Classical Field Theory?

It seems nowadays natural to deal with fields in Physics. It is pointed out here that the motivation to introduce fields, dynamical degrees of freedom distributed in space, is not supported only by electrodynamics. There is a "no-go" theorem in mechanics, it is impossible to construct relativistic interactions in a many-body system. Thus if special relativity is imposed we need an extension of the many-particle systems, such fields, to incorporate interactions.

The dynamical problem of a many-particle system is establishment and the solution of the equations of motion for the world lines $x^\mu_a(s)$, $a = 1m\ldots,n$ of the particles. By generalizing the Newton equation we seek differential equations for the world lines,

$$\ddot{x}^\mu_a = F^\mu_a(x_1,\ldots,x_n) \tag{3.1}$$

where interactions are described by some kind of "forces" $F^\mu_a(x_1,\ldots,x_n)$. The problem is that we intend to use instantaneous force and to consider the argument of the force, the world lines at the same time x^0_a as the particle in question but the "equal time" is not a relativistically invariant concept and has not natural implementation.

A formal aspect of this problem can be seen by recalling that $\dot{x}^2(s) = 1$ long the world line, therefore $\ddot{x}\cdot\dot{x} = 0$, the four-velocity and the four-acceleration are orthogonal. Thus any Cauchy problem which provides the initial coordinates and velocities on an initial spatial hyper-surface must satisfy this orthogonality constraint. This imposes a complicated, unexpected restriction on the possible forces. For instance when translation invariant, central two-particle forces are considered then

$$F_a^\mu(x_1,\ldots,x_n) = \sum_{b\neq a}(x_a^\mu - x_b^\mu)f((x_a-x_b)^2) \tag{3.2}$$

and $x_a - x_b$ is usually not orthogonal to $\dot{x}_a$ and x_b.

The most convincing and general proof of the "no-go" theorem is algebraic. The point is that the Hamilton function is the generator of the translation in time and its Poisson brackets, the commutator with the other generators of the Poincar group are fixed by the relativistic kinematics, the structure of the Poincar group. It can be proven that the any realization of the commutator algebra of the Poincar group for a many-particle system must contains the trivial Hamilton function, the sum of the free Hamilton functions for the particles.

What is left to introduce relativistic interactions is to give up instantaneous force and allow the influence of the whole past history of the system on the forces. This is an action-at-a-distance theory where particles interact at different space-time points. We can simplify this situation by introducing auxiliary dynamical variables which are distributed in space and describe the propagation of the influence of the particles on each other. The systematical implementation of this idea is classical field theory.

3.2 Variational principle

Our goal in Section is to obtain equations of motion which are local in space-time and are compatible with certain symmetries in a systematic manner. The basic principle is to construct equations which remain invariant under nonlinear transformations of the coordinates and the time. It is rather obvious that such a gigantic symmetry renders the resulting equations much more useful.

Field theory is a dynamical system containing degrees of freedom, denoted by $\phi(\mathbf{x})$, at each space point $\mathbf{x}$. The coordinate $\phi(\mathbf{x})$ can be a single real number (real scalar field) or consist n-components (n-component field). Our goal is to provide an equation satisfied by the trajectory $\phi_{cl}(t,\mathbf{x})$. The index *cl* is supposed to remind us that this trajectory is the solution of a classical (as opposed to a quantum) equation of motion.

This problem will be simplified in two steps. First we restrict $\mathbf{x}$ to a single value, $\mathbf{x} = \mathbf{x}_0$. The n-component field $\phi(\mathbf{x}_0)$ can be thought as the coordinate of a single point particle moving in n-dimensions. We need the equation satisfied by the trajectory of this particle. The second step of simplification is to reduce the n-dimensional function $\phi(\mathbf{x}_0)$ to a single point on the real axis.

3.2.1 Single point on the real axis

We start with a baby version of the dynamical problem, the identification of a point on the real axis, $x_{cl} \in \mathbb{R}$, in a manner which is independent of the re-parametrization of the real axis.

The solution is that the point is identified by specifying a function with vanishing derivative at x_{cl} only:

$$\frac{df(x)}{dx}_{|x=x_{cl}} = 0 \tag{3.3}$$

To check the re-parametrization invariance of this equation we introduce new coordinate y by the function $x = x(y)$ and find

$$\frac{df(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{df(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0 \tag{3.4}$$

We can now announce the variational principle. There is simple way of rewriting Eq. (3.3) by performing an infinitesimal variation of the coordinate $x \to x + \delta x$, and writing

$$\begin{aligned} f(x_{cl} + \delta x) &= f(x_{cl}) + \delta f(x_{cl}) \\ &= f(x_{cl}) + \delta x \underbrace{f'(x_{cl})}_{0} + \frac{\delta x^2}{2} f''(x_{cl}) + \mathcal{O}\left(\delta x^3\right). \end{aligned} \tag{3.5}$$

The variation principle, equivalent of Eq. (3.3) is

$$\delta f(x_{cl}) = \mathcal{O}\left(\delta x^2\right), \tag{3.6}$$

stating that x_{cl} is characterized by the property that an infinitesimal variation around it, $x_{cl} \to x_{cl} + \delta x$, induces an $\mathcal{O}\left(\delta x^2\right)$ change in the value of $f(x_{cl})$.

3.2.2 Non-relativistic point particle

We want to identify a trajectory of a non-relativistic particle in a coordinate choice independent manner.

Let us identify a trajectory $x_{cl}(t)$ by specifying the coordinate at the initial and final time, $x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f$ (by assuming that the equation of motion is of second order in time derivatives) and consider a variation of the trajectory $x(t)$: $x(t) \to x(t) + \delta x(t)$ which leaves the initial and final conditions invariant (ie. does not modify the solution). Our function $f(x)$ of the previous section becomes a functional, called action

$$S[x(\cdot)] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \tag{3.7}$$

involving the Lagrangian $L(x(t), \dot{x}(t))$. (The symbol $x(\cdot)$ in the argument of the action functional is supposed to remind us that the variable of the functional is a function. It is better to put a dot in the place of the independent variable of the function $x(t)$ otherwise the notation $S[x(t)]$ can be mistaken with an embedded function $S(x(t))$.) The variation of the action is

$$\begin{aligned}
\delta S[x(\cdot)] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \frac{d}{dt}\delta x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\
&= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\delta L(x(t), \dot{x}(t))}{\delta x} \\
&\quad + \frac{d}{dt}\delta x(t) \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\
&= \int_{t_i}^{t_f} dt \delta x(t) \left[\frac{\delta L(x(t), \dot{x}(t))}{\delta x} - \frac{d}{dt} \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} \right] \\
&\quad + \underbrace{\delta x(t)}_{0} \frac{\delta L(x(t), \dot{x}(t))}{\delta \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right)
\end{aligned} \tag{3.8}$$

The variational principle amounts to the suppression of the integral in the last line for an arbitrary variation, yielding the Euler-Lagrange equation:

$$\frac{\delta L(x, \dot{x})}{\delta x} - \frac{d}{dt} \frac{\delta L(x, \dot{x})}{\delta \dot{x}} = 0 \tag{3.9}$$

The generalization of the previous steps for a n-dimensional particle gives

$$\frac{\delta L(\mathbf{x}, \dot{\mathbf{x}})}{\delta \mathbf{x}} - \frac{d}{dt} \frac{\delta L(\mathbf{x}, \dot{\mathbf{x}})}{\delta \dot{\mathbf{x}}} = 0. \tag{3.10}$$

It is easy to check that the Lagrangian

$$L = T - U = \frac{m}{2}\dot{\mathbf{x}}^2 - U(\mathbf{x}) \tag{3.11}$$

leads to the usual Newton equation

$$m\ddot{\mathbf{x}} = -\nabla U(\mathbf{x}). \tag{3.12}$$

It is advantageous to introduce the generalized momentum:

$$p = \frac{\partial L(x,\dot{x})}{\partial \dot{x}} \tag{3.13}$$

which allows to write the Euler-Lagrange equation as

$$\dot{p} = \frac{\partial L(x,\dot{x})}{\partial x} \tag{3.14}$$

The coordinate not appearing in the Lagrangian in an explicit manner is called cyclic coordinate,

$$\frac{\partial L(x,\dot{x})}{\partial x_{cycl}} = 0. \tag{3.15}$$

For each cyclic coordinate there is a conserved quantity because the generalized momentum of a cyclic coordinate, p_{cycl} is conserved according to Eqs. (3.13) and (3.15).

3.2.3 Relativistic particle

After the heuristic generalization of the non-relativistic Newton's law let us consider now more systematically the relativistically invariant variational principle. The Lorentz invariant action must be proportional to the invariant length of the world-line, this latter being the only invariant of the problem. Dimensional considerations lead to

$$S = -mc\int_{s_i}^{s_f} ds = \int_{\tau_i}^{\tau_f} d\tau L_\tau \tag{3.16}$$

where τ is an arbitrary parameter of the world-line and the corresponding Lagrangian is

$$L_\tau = -mc\sqrt{\frac{dx^\mu}{d\tau}g_{\mu\nu}\frac{dx^\mu}{d\tau}}. \tag{3.17}$$

The Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} = -mc^2 + \frac{v^2}{2m} + \mathcal{O}\left(\frac{v^4}{c^2}\right) \tag{3.18}$$

corresponds to the integrand when τ is the time and justifies the dimensionless constant in the definition of the action (3.16).

We have immediately the energy-momentum

$$\begin{aligned}
\mathbf{p} &= \frac{\partial L}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \\
E &= \vec{p}\vec{v} - L = \frac{mc^2}{\sqrt{1-\frac{v^2}{c}}} = mc^2 + \frac{v^2}{2m} + \mathcal{O}\left(\frac{v^4}{c^2}\right).
\end{aligned} \tag{3.19}$$

The variation of the world-line,

$$\begin{aligned}
\delta S &= \int_{x_i}^{x_f} ds \left(\frac{\delta L_s}{\delta x^\mu}\delta x^\mu + \frac{\delta L_s}{\delta \frac{dx^\mu}{ds}}\delta \frac{dx^\mu}{ds}\right) \\
&= \frac{\delta L_s}{\delta \frac{dx^\mu}{ds}}\delta x^\mu \Big|_{x_i}^{x_f} + \int_{x_i}^{x_f} ds\delta x^\mu \left(\frac{\delta L_s}{\delta x^\mu} - \frac{d}{ds}\frac{\delta L_s}{\delta \frac{dx^\mu}{ds}}\right)
\end{aligned} \tag{3.20}$$

or

$$\begin{aligned}
\delta S &= -mc\int ds \frac{\frac{\delta dx^\mu}{ds}\frac{dx_\mu}{ds}}{\sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}} \\
&= -mc\int ds \frac{\delta dx^\mu}{ds}\frac{dx_\mu}{ds} \\
&= -mc\delta x^\mu \frac{dx_\mu}{ds}\Big|_{x_i}^{x_f} + mc\int ds\delta x^\mu \frac{d^2 x_\mu}{ds^2}
\end{aligned} \tag{3.21}$$

leads to the Euler-Lagrange equation

$$mc\frac{d^2x^\mu}{ds^2} = 0. \tag{3.22}$$

The four momentum is

$$p_\mu = -\frac{\delta S}{\delta x^\mu_f} = mcg_{\mu\nu}\frac{dx^\nu}{ds}. \tag{3.23}$$

The projection of the non-relativistic angular momentum on a given unit vector $\mathbf{n}$ can be defined by the derivative of the action with respect to the angle of rotation around $\mathbf{n}$. Such a rotation generates $\delta\mathbf{x} = \delta R\mathbf{x} = \delta\phi\mathbf{n}\times\mathbf{x}$ and gives

$$\frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta x^\ell_f}\frac{\delta x^\ell}{\delta\phi} = \mathbf{p}R\mathbf{x} = \mathbf{p}(\mathbf{n}\times\mathbf{x}) = \mathbf{n}(\mathbf{x}\times\mathbf{p}). \tag{3.24}$$

The relativistic generalization of this procedure is $\delta x_\mu = \delta L_{\mu\nu} x^\nu$,

$$\frac{\delta S}{\delta \phi} = \frac{\delta S}{\delta x^\rho}\frac{\delta x^\rho}{\delta \phi} = -p^\mu L_{\mu\nu} x^\nu = \frac{1}{2} L_{\mu\nu}(p^\nu x^\mu - p^\mu x^\nu) \tag{3.25}$$

yielding

$$M^{\mu\nu} = x^\mu p^\nu - p^\mu x^\nu. \tag{3.26}$$

3.2.4 Scalar field

We turn now the dynamical variables which were evoked in avoiding the "no-go" theorem, fields. We assume the simple case where there are n scalar degree of freedom at each space point, a scalar field $\phi_a(\mathbf{x})$, $a = 1, \ldots, n$ whose time dependence gives a space-time dependent field $\phi_a(x)$.

To establish the variational principle we consider the variation of the trajectory $\phi(x)$

$$\phi(x) \to \phi(x) + \delta\phi(x), \qquad \delta\phi(t_i, \mathbf{x}) = \delta\phi(t_f, \mathbf{x}) = 0. \tag{3.27}$$

The variation of the action

$$S[\phi(\cdot)] = \int_V \underbrace{dt d^3x}_{dx} L(\phi, \partial\phi) \tag{3.28}$$

is

$$\begin{aligned} \delta S &= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial \phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial \partial_\mu \phi_a}\delta\partial_\mu\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\ &= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial \phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial \partial_\mu \phi_a}\partial_\mu\delta\phi_a \right) + \mathcal{O}\left(\delta^2\phi\right) \\ &= \int_{\partial V} ds^\mu \delta\phi_a \frac{\partial L(\phi,\partial\phi)}{\partial \partial_\mu \phi_a} \\ &\quad + \int_V dx \delta\phi_a \left(\frac{\partial L(\phi,\partial\phi)}{\partial \phi_a} - \partial_\mu \frac{\partial L(\phi,\partial\phi)}{\partial \partial_\mu \phi_a} \right) + \mathcal{O}\left(\delta^2\phi\right) \end{aligned} \tag{3.29}$$

The first term for $\mu = 0$,

$$\begin{aligned} \int_{\partial V} ds^0 \delta\phi_a \frac{\partial L(\phi,\partial\phi)}{\partial \partial_0 \phi_a} &= \int_{t=t_f} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi,\partial\phi)}{\partial \partial_0 \phi_a} \\ &\quad - \int_{t=t_i} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi,\partial\phi)}{\partial \partial_0 \phi_a} = 0 \end{aligned} \tag{3.30}$$

is vanishing because there is no variation at the initial and final time. When $\mu = j$ then

$$\int_{\partial V} ds^j \delta\phi_a \frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a} = \int_{x_j=\infty} ds^j \delta\phi_a \underbrace{\frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a}}_{0} - \int_{x_j=-\infty} ds^j \delta\phi_a \underbrace{\frac{\partial L(\phi, \partial\phi)}{\partial\partial_j\phi_a}}_{0} = 0 \quad (3.31)$$

and it is still vanishing because we are interested in the dynamics of localized systems and the interactions are supposed to be short ranged. Therefore, $\phi = 0$ at the spatial infinities and the Lagrangian is vanishing. The suppression of the second term gives the Euler-Lagrange equation

$$\frac{\partial L(\phi, \partial\phi)}{\partial\phi_a} - \partial_\mu \frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi_a} = 0. \quad (3.32)$$

Let us consider a scalar field as an example. The four momentum is represented by the vector operator $\hat{p}_\mu = -\left(\frac{\hbar}{ic}\partial_0, \frac{\hbar}{i}\vec{\partial}\right)$ in Quantum Mechanics which leads to the Lorentz invariant invariant Klein-Gordon equation

$$0 = (\hat{p}^2 - m^2c^2)\phi_a = -\hbar^2\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi_a, \quad (3.33)$$

generated by the Lagrangian

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{m^2c^2}{2\hbar^2}\phi^2 \implies \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2. \quad (3.34)$$

The parameter m can be interpreted as mass because the plane wave solution

$$\phi_k(x) = e^{-ik\cdot x} \quad (3.35)$$

to the equation of motion satisfies the mass shell condition,

$$\hbar^2k^2 = m^2c^2 \quad (3.36)$$

c.f. Eq. (2.43).

One may introduce a relativistically invariant self-interaction by means of a potential $V(\phi)$,

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - V(\phi) \quad (3.37)$$

and the corresponding equation of motion is

$$(\partial_\mu\partial^\mu + m^2) = -V'(\phi). \quad (3.38)$$

3.3 Noether theorem

It is shown below that there is a conserved current for each continuous symmetry.

Symmetry: A transformation of the space-time coordinates $x^\mu \to x'^\mu$, and the field $\phi_a(x) \to \phi'_a(x)$ preserves the equation of motion. Since the equation of motion is obtained by varying the action, the action should be preserved by the symmetry transformations. A slight generalization is that the action can in fact be changed by a surface term which does not influence its variation, the equation of motion at finite space-time points. Therefore, the symmetry transformations satisfy the condition

$$L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu \Lambda^\mu \tag{3.39}$$

with a certain vector function $\Lambda^\mu(x')$.

Continuous symmetry: There are infinitesimal symmetry transformations, in an arbitrary small neighborhood of the identity, $x^\mu \to x^\mu + \delta x^\mu$, $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$. Examples: Rotations, translations in the space-time, and $\phi(x) \to e^{i\alpha}\phi(x)$ for a complex field.

Conserved current: $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \partial v j = -\int_{\partial V} d\mathbf{s} \cdot \mathbf{j} \tag{3.40}$$

It is useful to distinguish external and internal spaces, corresponding to the space-time and the values of the field variable. Eg.

$$\phi_a(x) : \underbrace{\mathbb{R}^4}_{\text{external space}} \to \underbrace{\mathbb{R}^m}_{\text{internal space}} . \tag{3.41}$$

Internal and external symmetry transformations act on the internal or external space, respectively.

3.3.1 Point particle

The main points of the construction of the Noether current for internal symmetries can be best understood in the framework of a particle. To find the analogy of the internal symmetries let us consider a point particle with the continuous symmetry $\mathbf{x} \to \mathbf{x} + \epsilon\mathbf{f}(\mathbf{x})$ for infinitesimal ϵ,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = L(\mathbf{x} + \epsilon\mathbf{f}(\mathbf{x}), \dot{\mathbf{x}} + \epsilon(\dot{\mathbf{x}} \cdot \partial)\mathbf{f}(\mathbf{x})) + \mathcal{O}\left(\epsilon^2\right). \tag{3.42}$$

Let us introduce a new, time dependent coordinates, $\mathbf{y}(t) = \mathbf{y}(\mathbf{x}(t))$, based on the solution of the equation of motion, $\mathbf{x}_{cl}(t)$, in such a manner that one of them will be $y^1(t) = \epsilon(t)$, where $\mathbf{x}(t) = \mathbf{x}_{cl}(t) + \epsilon(t)\mathbf{f}(\mathbf{x}_{cl}(t))$. There will be $n-1$ other new coordinates, y^ℓ, $\ell = 2, \ldots, n$ whose actual form is not interesting for us. The Lagrangian in terms of the new coordinates is defined by $L(\mathbf{y}, \dot{\mathbf{y}}) = L(\mathbf{y}(\mathbf{x}), \dot{\mathbf{y}}(\mathbf{x}))$. The ϵ-dependent part assumes the form

$$L(\epsilon, \dot\epsilon) = L(\mathbf{x}_{cl} + \epsilon\mathbf{f}(\mathbf{x}_{cl}), \dot{\mathbf{x}}_{cl} + \epsilon(\dot{\mathbf{x}}_{cl}\cdot\partial)\mathbf{f}(\mathbf{x}_{cl}) + \dot\epsilon\mathbf{f}(\mathbf{x}_{cl})) + \mathcal{O}\left(\epsilon^2\right). \quad (3.43)$$

What is the equation of motion of this Lagrangian? Since the solution is $\epsilon(t) = 0$ it is sufficient to retain the $\mathcal{O}(\epsilon)$ contributions in the Lagrangian only,

$$\begin{aligned} L(\epsilon, \dot\epsilon) \to L^{(1)}(\epsilon, \dot\epsilon) &= \epsilon\frac{\partial L(\mathbf{x}_{cl}, \dot{\mathbf{x}}_{cl})}{\partial\mathbf{x}}\cdot\mathbf{f}(\mathbf{x}_{cl}) \\ &+\frac{\partial L(\mathbf{x}_{cl}, \dot{\mathbf{x}}_{cl})}{\partial\dot{\mathbf{x}}}[\epsilon(\dot{\mathbf{x}}_{cl}\cdot\partial)\mathbf{f}(\mathbf{x}_{cl}) + \dot\epsilon\mathbf{f}(\mathbf{x}_{cl})] \quad (3.44) \end{aligned}$$

up to an ϵ-independent constant. The corresponding Euler-Lagrange equation is

$$\frac{\partial L^{(1)}(\epsilon, \dot\epsilon)}{\partial\epsilon} - \frac{d}{dt}\frac{\partial L^{(1)}(\epsilon, \dot\epsilon)}{\partial\dot\epsilon} = 0. \quad (3.45)$$

(this is the point where the formal invariance of the equation of motion under nonlinear, time dependent transformations of the coordinates is used). According to Eq. (3.42) ϵ is a cyclic coordinate,

$$\frac{\partial L(\epsilon, \dot\epsilon)}{\partial\epsilon} = 0 \quad (3.46)$$

and its generalized momentum,

$$p_\epsilon = \frac{\partial L(\epsilon, \dot\epsilon)}{\partial\dot\epsilon} \quad (3.47)$$

is conserved.

The external space transformation corresponds to the shift of the time, $t \to t + \epsilon$ which induces $x(t) \to x(t-\epsilon) = x(t) - \epsilon\dot{x}(t)$ for infinitesimal ϵ. This is a symmetry as long as the Hamiltonian (and the Lagrangian) does not contain explicitly the time. In fact, the action changes by a boundary contribution only which can be seen by expanding the Lagrangian in time around $t - \epsilon$,

$$\int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) = \int_{t_i}^{t_f} dt \left[L(x(t-\epsilon), \dot{x}(t-\epsilon)) + \epsilon\frac{dL(x(t), \dot{x}(t))}{dt}\right] \quad (3.48)$$

up to $\mathcal{O}\left(\epsilon^2\right)$ terms and as a result the variational equation of motion remains unchanged. But the continuation of the argument is slightly different from the case of internal symmetry. We consider ϵ as a time dependent function which generates a transformation of the coordinate, $x(t) \to x(t-\epsilon(t)) = x(t) - \epsilon(t)\dot{x}(t) + \mathcal{O}\left(\epsilon^2\right)$. The Lagrangian of $\epsilon(t)$ as new coordinate for the choice $x(t) = x_{cl}(t)$ is

$$
\begin{aligned}
\tilde{L}(\epsilon,\dot{\epsilon}) &= L(x_{cl}(t-\epsilon),\dot{x}_{cl}(t-\epsilon)) - L(x_{cl}(t),\dot{x}_{cl}(t)) \\
&= -\epsilon\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial x} - \frac{d\epsilon\dot{x}_{cl}}{dt}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}} \\
&= \underbrace{-\epsilon\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial x} - \epsilon\ddot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}}}_{-\epsilon\frac{dL(x_{cl},\dot{x}_{cl})}{dt}} - \dot{\epsilon}\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}} \\
&= -\epsilon\left[\frac{dL(x_{cl},\dot{x}_{cl})}{dt} - \frac{d}{dt}\left(\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}}\dot{x}_{cl}\right)\right] \\
&\quad - \frac{d}{dt}\left(\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}_{cl}}\epsilon\dot{x}_{cl}\right) \qquad (3.49)
\end{aligned}
$$

up to an ϵ-independent constant and $\mathcal{O}\left(\epsilon^2\right)$ contributions and its equation of motion, Eq. (3.45), assures the conservation of the energy,

$$H = \frac{\partial L(x,\dot{x})}{\partial \dot{x}}\dot{x} - L(x,\dot{x}). \qquad (3.50)$$

3.3.2 Internal symmetries

An internal symmetry transformation of field theory acts on the internal space only. We shall consider linearly realized internal symmetries for simplicity where

$$\delta x^\mu = 0, \quad \delta_i\phi_a(x) = \epsilon \underbrace{\tau_{ab}}_{generator} \phi_b(x). \qquad (3.51)$$

This transformation is a symmetry,

$$L(\phi,\partial\phi) = L(\phi+\epsilon\tau\phi,\partial\phi+\epsilon\tau\partial\phi) + \mathcal{O}\left(\epsilon^2\right). \qquad (3.52)$$

Let us introduce new "coordinates", ie. new field variable, $\Phi(\phi)$, in such a manner that $\Phi^1(x) = \epsilon(x)$ where $\phi(x) = \phi_{cl}(x) + \epsilon(x)\tau\phi_{cl}(x)$, $\phi_{cl}(x)$ being the solution of the equations of movement. The linearized Lagrangian for $\epsilon(x)$ is

$$
\begin{aligned}
\tilde{L}(\epsilon,\partial\epsilon) &= L(\phi_{cl}+\epsilon\tau\phi(x),\partial\phi_{cl}+\partial\epsilon\tau\phi(x)+\epsilon\tau\partial\phi(x)) \\
&\to \epsilon\tau\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\phi} + [\partial\epsilon\tau\phi(x)+\epsilon\tau\partial\phi(x)]\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\partial\phi} \qquad (3.53)
\end{aligned}
$$

The symmetry, Eq. (3.52), indicates that ϵ is a cyclic coordinate and the equation of motion

$$\frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\epsilon} - \partial_\mu \frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} = 0. \tag{3.54}$$

shows that the current,

$$J^\mu = -\frac{\partial \tilde{L}(\epsilon, \partial\epsilon)}{\partial\partial_\mu\epsilon} = -\frac{\partial L(\phi, \partial\phi)}{\partial\partial_\mu\phi}\tau\phi \tag{3.55}$$

defined up to a multiplicative constant as the generalized momentum of ϵ, is conserved. Notice that (i) we have an independent conserved current corresponding to each independent direction in the internal symmetry group and (ii) the conserved current is well defined up to a multiplicative constant only.

Let us consider a complex scalar field with symmetry $\phi(x) \to e^{i\alpha}\phi(x)$ as an example. The theory is defined by the Lagrangian

$$L = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V(\phi^*\phi) \tag{3.56}$$

where it is useful to considered ϕ and ϕ^* as independent variables. The infinitesimal transformations $\delta\phi = i\epsilon\phi$, $\delta\phi^* = -i\epsilon\phi^*$ yield the conserved current

$$j^\mu = \frac{i}{2}(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi) \tag{3.57}$$

up to a multiplicative constant.

3.3.3 External symmetries

The most general transformations leaving the action invariant may act in the external space, too. Therefore, let us consider the transformation $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$ and $\phi(x) \to \phi'(x') = \phi(x) + \delta\phi(x)$ where $\delta\phi(x) = \delta_i\phi(x) + \delta x^\mu\partial_\mu\phi(x)$ where $\delta_i\phi(x)$ denotes the eventual internal space variation. The variation of the action is

$$\begin{aligned}\delta S &= \int_V dx\delta L + \int_{V'-V} dxL \\ &= \int_V dx\delta L + \int_{\partial V} dS_\mu\delta x^\mu L \end{aligned} \tag{3.58}$$

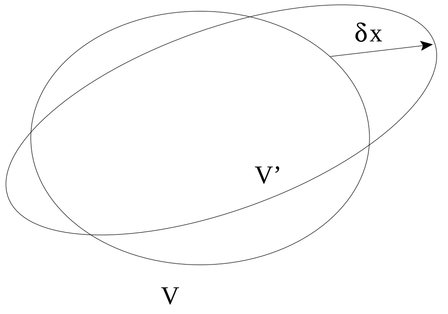

Figure 3.1: Deformation of the volume in the external space.

according to Fig. 3.1 what can be written as

$$\begin{aligned}\delta S &= \int_V dx \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi}\right) \delta\phi + \int_{\partial V} dS_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi}\delta_i \phi + \delta x^\mu L\right) \\ &= \int_V dx \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi}\right) \delta\phi \\ &\quad + \int_{\partial V} dS_\mu \left[\frac{\partial L}{\partial \partial_\mu \phi}\delta\phi + \delta x^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi}\partial_\nu \phi\right)\right]. \end{aligned} \tag{3.59}$$

For field configurations satisfying the equation of motion the first integral is vanishing leaving the current

$$J^\mu = \frac{\partial L}{\partial \partial_\mu \phi}\delta\phi + \delta x^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi}\partial_\nu \phi\right) \tag{3.60}$$

conserved.

The case of internal space variation only $\delta x^\mu = 0$ reproduces the conserved Noether current of Eq. (3.55). For translations we have $\delta x^\mu = a^\mu$ and $\delta_i \phi = 0$ is chosen such that the field configuration is displaced only, $\delta\phi = 0$. The four conserved current are collected in the canonical energy-momentum tensor

$$T_c^{\mu\nu} = \frac{\partial L}{\partial \partial_\mu \phi}\partial^\nu \phi - L g^{\mu\nu} \tag{3.61}$$

obeying the conservation laws

$$\partial_\mu T_c^{\mu\nu} = 0. \tag{3.62}$$

They show that

$$P^\nu = \int d^3x T_c^{0\nu} \tag{3.63}$$

can be identified by the energy-momentum vector and we have the form

$$T_c^{\mu\nu} = \begin{pmatrix} \epsilon & c\mathbf{p} \\ \frac{1}{c}\mathbf{S} & \sigma \end{pmatrix} \tag{3.64}$$

where ϵ is the energy density, $\mathbf{p}$ is the momentum density, $\mathbf{S}$ is the density of the energy flux and σ^{jk} is the flux of p^k in the direction j.

When Lorentz transformations and translations are performed simultaneously then we have $\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu$ and $\delta\phi = \Lambda^{\nu\mu}\omega_{\mu\nu}\phi \neq 0$ for field with nonvanishing spin and the conserved current is

$$J^\mu = \frac{\partial L}{\partial\partial_\mu\phi}(\Lambda^{\nu\kappa}\omega_{\kappa\nu}\phi - \delta x^\nu\partial_\nu\phi) + \delta x^\mu L. \tag{3.65}$$

Let us simplify the expressions be introducing the tensor

$$f^{\mu\nu\kappa} = \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa}\phi \tag{3.66}$$

and write

$$J^\mu = f^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L. \tag{3.67}$$

By the cyclic permutation of the indices $\mu\nu\kappa$ we can define another tensor

$$\tilde{f}^{\mu\nu\kappa} = \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi \tag{3.68}$$

which is antisymmetric in the first two indices,

$$\begin{aligned}\tilde{f}^{\nu\mu\kappa} &= \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\kappa\nu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\nu\mu}\right)\phi \\ &= \left(-\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi \\ &= -\tilde{f}^{\mu\nu\kappa}\end{aligned} \tag{3.69}$$

and verifies the equation

$$\begin{aligned}\tilde{f}^{\mu\nu\kappa}\omega_{\nu\kappa} &= \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa} - \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa}.\end{aligned} \tag{3.70}$$

As a result we can replace $f^{\mu\nu\kappa}$ by it in Eq. (3.67),

$$\begin{aligned} J^\mu &= \tilde{f}^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \tilde{f}^{\mu\nu\kappa}\partial_\nu(\delta x_\kappa) - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \delta x_\kappa\left(g^{\mu\kappa}L - \frac{\partial L}{\partial\partial_\mu\phi}\partial^\kappa\phi - \partial_\nu\tilde{f}^{\mu\nu\kappa}\right) + \partial_\nu(\tilde{f}^{\mu\nu\kappa}\delta x_\kappa\phi). \end{aligned} \tag{3.71}$$

The last term $J'^\mu = \partial_\nu(\tilde{f}^{\mu\nu\kappa}\delta x_\kappa\phi)$ gives a conserved current thus can be dropped and the conserved Noether current simplifies as

$$J^\mu = T^{\mu\nu}(a_\nu + \omega_{\nu\kappa}x^\kappa) = T^{\mu\nu}a_\nu + \frac{1}{2}(T^{\mu\nu}x^\kappa - T^{\mu\kappa}x^\nu)\omega_{\nu\kappa} \tag{3.72}$$

where we can introduced the symmetric energy momentum tensor

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\kappa\tilde{f}^{\mu\kappa\nu} \tag{3.73}$$

and the tensor

$$M^{\mu\nu\sigma} = T^{\mu\nu}x^\sigma - T^{\mu\sigma}x^\nu. \tag{3.74}$$

Due to

$$\int_{\partial V} S_\mu\partial_\kappa\tilde{f}^{\mu\kappa\nu} = \int_V \partial_\mu\partial_\kappa\tilde{f}^{\mu\kappa\nu} = 0 \tag{3.75}$$

the energy momentum extracted from $T^{\mu\nu}$ and $T_c^{\mu\nu}$ agree and M is conserved

$$\partial_\mu M^{\mu\nu\sigma} = 0, \tag{3.76}$$

yielding the relativistic angular momentum

$$J^{\nu\sigma} = \int d^3x(T^{0\nu}x^\sigma - T^{0\sigma}x^\nu). \tag{3.77}$$

with the usual non-relativistic spatial structure. The energy-momentum tensor $T^{\mu\nu}$ is symmetric because the conservation of the relativistic angular momentum, Eq. (3.76) gives

$$0 = \partial_\rho M^{\rho\mu\nu} = \partial_\rho(T^{\rho\mu}x^\nu - T^{\rho\nu}x^\mu) = T^{\nu\mu} - T^{\mu\nu}. \tag{3.78}$$

Chapter 4

Electrodynamics

4.1 Charge in an external electromagnetic field

The three-dimensional scalar and vector fields make up the four-dimensional vector potential as $A^\mu = (\phi, \mathbf{A})$ and the simplest Lorentz invariant Lagrange function we can construct with it is $A_\mu \dot{x}^\mu$ therefore the action for a point-charge moving in the presence of a given, external vector potential is

$$\begin{aligned} S &= -\int_{x_i}^{x_f} \left(mcds + \frac{e}{c} A_\mu dx^\mu \right) \\ &= -\int_{x_i}^{x_f} \left(mcds - \frac{e}{c} \mathbf{A} \cdot d\mathbf{x} + e\phi dt \right) \\ &= \int_{\tau_i}^{\tau_f} L_\tau d\tau, \end{aligned} \tag{4.1}$$

where the index τ in the Lagrangian is a reminder of the variable used to construct the action,

$$L_t = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - e\phi, \tag{4.2}$$

or

$$L_s = -mc \sqrt{\frac{dx^\mu}{ds} g_{\mu\nu} \frac{dx^\nu}{ds}} - \frac{e}{c} A_\mu(x) \frac{dx^\mu}{ds}. \tag{4.3}$$

The Euler-Lagrange equation for the manifest invariant L_s which is parametrized by the invariant length s of the world line is

$$\begin{aligned} 0 &= \frac{\delta L}{\delta x^\mu} - \frac{d}{ds}\frac{\delta L}{\delta \frac{dx^\mu}{ds}} \\ &= -\frac{e}{c}\partial_\mu A_\nu(x)\frac{dx^\nu}{ds} + mc\frac{d}{ds}\frac{g_{\mu\nu}\frac{dx^\mu}{ds}}{\sqrt{\frac{dx^\mu}{ds}g_{\mu\nu}\frac{dx^\nu}{ds}}} + \frac{e}{c}\frac{d}{ds}A_\mu(x) \\ &= mc\frac{d^2x_\mu}{ds^2} - \frac{e}{c}F_{\mu\nu}\frac{dx^\nu}{ds} \end{aligned} \tag{4.4}$$

where the field-strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{4.5}$$

The interaction term in the action can be written as a space-time integral involving the current density,

$$S = -mc\int ds - \frac{1}{c}\int dx A_\mu(x) j^\mu(x). \tag{4.6}$$

The relativistically covariant generalization of the non-relativistic current $\mathbf{j} = \rho\mathbf{v}$ for a single charge is

$$j^\mu = \rho\frac{dx^\mu}{dt} = (c\rho, \mathbf{j}) = (c\rho, \rho\mathbf{v}) = \rho\frac{ds}{dt}\dot{x}^\mu \tag{4.7}$$

In the case of a system of charges, $\mathbf{x}_a(t)$, we have

$$\begin{aligned} j^\mu(x) &= c\sum_a e_a \int ds\delta(x - x_a(s))\dot{x}^\mu \\ &= c\sum_a e_a \int ds\delta(\mathbf{x} - \mathbf{x}_a(s))\delta(x^0 - x_a^0(s))\dot{x}^\mu \\ &= c\sum_a e_a\delta(\mathbf{x} - \mathbf{x}_a(s))\frac{1}{|\frac{dx^0}{ds}|}\dot{x}^\mu \\ &= \underbrace{\sum_a e_a\delta(\mathbf{x} - \mathbf{x}_a(s))}_{\rho(\mathbf{x})}\frac{dx^\mu}{dt}. \end{aligned} \tag{4.8}$$

It is easy to verify that the continuity equation

$$\begin{aligned} \partial_\mu j^\mu &= \partial_0\rho + \nabla\cdot\mathbf{j} \\ &= \sum_a e_a[-\mathbf{v}_a(t)\nabla\delta(\mathbf{x} - \mathbf{x}_a(t)) + \nabla\delta(\mathbf{x} - \mathbf{x}_a(t))\mathbf{v}_a(t)] = 0 \end{aligned} \tag{4.9}$$

is satisfied.

4.2 Dynamics of the electromagnetic field

The action (4.6) dos not contain the time derivatives of the vector potential therefore we have to extend our Lagrangian, $L \to L + L_A$, to generate dynamics for the electromagnetic field. The guiding principle is that L_A should be

1. quadratic in the time derivative of the vector potential to have the usual equation of motion,
2. Lorentz invariant and
3. gauge invariant, ie. remain invariant under the transformation

$$A_\mu \to A_\mu + \partial_\mu \alpha. \tag{4.10}$$

The simplest solution is

$$L_A = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \tag{4.11}$$

where the factor $-1/16\pi$ is introduced for later convenience. The complete action is $S = S_m + S_A$ where

$$S_m = -mc \sum_a \int ds \sqrt{\frac{dx_a^\mu}{ds} g_{\mu\nu} \frac{dx_a^\nu}{ds}} \tag{4.12}$$

and

$$\begin{aligned} S_A &= -\frac{e}{c} \sum_a \int A_\mu(x) dx^\mu - \frac{1}{16\pi c} \int F^{\mu\nu} F_{\mu\nu} dx \\ &= -\frac{e}{c} \sum_a \int \delta^{(3)}(\mathbf{x} - \mathbf{x}_a(t)) A_\mu(x) dx_a^\mu dV - \frac{1}{16\pi c} \int F^{\mu\nu} F_{\mu\nu} dx \\ &= -\frac{e}{c^2} \sum_a \int \delta^{(4)}(x - x_a(t)) A_\mu(x) \frac{dx^\mu}{dt} dx - \frac{1}{16\pi c} \int F^{\mu\nu} F_{\mu\nu} dx \\ &= \int L_A dV dt \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} L_A &= -\frac{1}{c} j^\mu A_\mu(x) - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \\ &= -\frac{1}{c} j^\mu A_\mu(x) - \frac{1}{8\pi} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{8\pi} \partial_\mu A_\nu \partial^\nu A^\mu. \end{aligned} \tag{4.14}$$

It yields the Maxwell-equations

$$0 = \frac{\delta L}{\delta A_\mu} - \partial_\nu \frac{\delta L}{\delta \partial_\nu A_\mu} = -\frac{1}{c} j^\mu - \frac{1}{4\pi} \partial_\nu F^{\mu\nu}. \tag{4.15}$$

Note that the necessary condition for the gauge invariance of the action is the current conservation, Eq. (4.9).

A simple calculation shows that any continuously double differentiable vector potential satisfies the Bianchi identity,

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0. \tag{4.16}$$

The usual three-dimensional notation is achieved by the parametrization $A^\mu = (\phi, \mathbf{A})$, $A_\mu = (\phi, -\mathbf{A})$, giving the electric and the magnetic fields

$$\begin{aligned} \mathbf{E} &= -\partial_0 \mathbf{A} - \nabla\phi = -\frac{1}{c}\partial_t \mathbf{A} - \nabla\phi, \\ \mathbf{H} &= \nabla \times \mathbf{A}. \end{aligned} \tag{4.17}$$

Notice that transformation $j^\mu = (\rho, \mathbf{j}) \to (\rho, -\mathbf{j})$ under time reversal and the invariance of the term $j^\mu A_\mu$ interaction Lagrangian requires the transformation law $\phi \to \phi$, $\mathbf{A} \to \mathbf{A}$, $\mathbf{E} \to \mathbf{E}$, $\mathbf{H} \to -\mathbf{H}$ for time reversal. The equation

$$\epsilon_{jk\ell} H_\ell = \epsilon_{jk\ell}\epsilon_{\ell mn} \nabla_m A_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\nabla_m A_n = \nabla_j A_k - \nabla_k A_j \tag{4.18}$$

relates the electric and magnetic field with the field strength tensor as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}. \tag{4.19}$$

One defines the dual field strength as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{4.20}$$

Duality refers to the exchange of the electric and the magnetic fields up to a sign,

$$\tilde{F}_{0j} = \frac{1}{2}\epsilon_{jk\ell} F^{k\ell} = B_j, \qquad \tilde{F}_{jk} = \epsilon_{jk\ell} F^{\ell 0} = -\epsilon_{jk\ell} E_\ell, \tag{4.21}$$

giving

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{4.22}$$

We have two invariants,

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= -2\mathbf{E}^2 + 2\mathbf{H}^2 \\ F^{\mu\nu}\tilde{F}_{\mu\nu} &= -4\mathbf{EH} \end{aligned} \tag{4.23}$$

but the first can be used only in classical electrodynamics which is invariant under time reversal. The field strength tensor transforms under Lorentz transformations as

$$\phi = \frac{\phi' + \frac{v}{c}A'_{\|}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad A_{\|} = \frac{A'_{\|} + \frac{v}{c}\phi'}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{4.24}$$

and

$$\begin{aligned} F^{\perp\perp} &= F^{\perp\perp'} \\ F^{\|\perp} &= \frac{F^{\|\perp'} + \frac{v}{c}F^{0\perp'}}{\sqrt{1-\frac{v^2}{c^2}}} \\ F^{0\perp} &= \frac{F^{0\perp'} + \frac{v}{c}F^{\|\perp'}}{\sqrt{1-\frac{v^2}{c^2}}} \\ F^{\|0} &= F^{\|0'} \ (\sim \epsilon^{01}). \end{aligned} \tag{4.25}$$

For $\mathbf{v} = (v, 0, 0)$ we have in the three-dimensional notation

$$\begin{aligned} E_{\|} &= E'_{\|}, \quad E_y = \frac{E'_y + \frac{v}{c}H'_z}{\sqrt{1-\frac{v^2}{c^2}}}, \quad E_z = \frac{E'_z - \frac{v}{c}H'_y}{\sqrt{1-\frac{v^2}{c^2}}} \\ H_{\|} &= H'_{\|}, \quad H_y = \frac{H'_y - \frac{v}{c}E'_z}{\sqrt{1-\frac{v^2}{c^2}}}, \quad H_z = \frac{H'_z + \frac{v}{c}E'_y}{\sqrt{1-\frac{v^2}{c^2}}}, \end{aligned} \tag{4.26}$$

i.e. the homogeneous electric and magnetic fields transform into each other when seen by an observer moving with constant speed.

4.3 Energy-momentum tensor

Let us first construct the energy-momentum tensor for the electromagnetic field by means of the Noether theorem. The translation $x^\mu \to x^\mu + \epsilon^\mu$ is a symmetry of the dynamics therefore we have a conserved current for each space-time direction, $(J^\mu)^\nu$ which can be rearranged in a tensor, $T^{\mu\nu} = (J^\mu)^\nu$, given by

$$T_c^{\mu\nu} = -g^{\mu\nu} L + \frac{\delta L}{\delta \partial_\mu A_\rho} \partial^\nu A_\rho = g^{\mu\nu} \left(\frac{1}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{c} j^\rho A_\rho \right) - \frac{1}{4\pi} F^{\mu\rho} \partial^\nu A_\rho \tag{4.27}$$

for the canonical energy-momentum tensor. The conservation law, $\partial_\mu T_c^{\mu\nu} = 0$ suggests the identification of $T_c^{0\nu}$ with the energy-momentum P^ν of the system up to a multiplicative constant. But the physical energy-momentum may contain a freely chosen three index tensor $\Theta^{\mu\rho\nu}$ as long as $\Theta^{\mu\rho\nu} = -\Theta^{\rho\mu\nu}$ because

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho \Theta^{\mu\rho\nu} \tag{4.28}$$

is still conserved. This freedom can be used to eliminate an unphysical property of the canonical energy-momentum tensor, namely its gauge dependence. The choice $\Theta^{\mu\rho\nu} = \frac{1}{4\pi} F^{\mu\rho} A^\nu$ gives

$$\begin{aligned}
T^{\mu\nu} &= g^{\mu\nu} \left(\frac{1}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{c} j^\rho A_\rho \right) - \frac{1}{4\pi} F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4\pi} \partial_\rho (F^{\mu\rho} A^\nu) \\
&= \frac{g^{\mu\nu}}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{4\pi} F^{\mu\rho} F_\rho{}^\nu + g^{\mu\nu} \frac{1}{c} j^\rho A_\rho + \frac{1}{4\pi} \partial_\rho F^{\mu\rho} A^\nu \\
&= \frac{g^{\mu\nu}}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{4\pi} F^{\mu\rho} F_\rho{}^\nu + g^{\mu\nu} \frac{1}{c} j^\rho A_\rho - j^\mu A^\nu
\end{aligned} \tag{4.29}$$

where the equation of motion was used in the last equation. The new energy-momentum tensor in the absence of the electric current, the true energy-momentum tensor of the EM field,

$$T_{ed}^{\mu\nu} = \frac{g^{\mu\nu}}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{4\pi} F^{\mu\rho} F_\rho{}^\nu, \tag{4.30}$$

is gauge invariant, symmetric and traceless. But it is not conserved, the energy-momentum is continuously exchanged between the charges and the EM field. The amount of non-conservation, $K^\nu = -\partial_\mu T_{ed}^{\mu\nu} \neq 0$, identifies the energy-momentum density of the charges,

$$\begin{aligned}
K^\nu &= -\partial_\mu \left(\frac{g^{\mu\nu}}{16\pi} F^{\rho\sigma} F_{\rho\sigma} + \frac{1}{4\pi} F^{\mu\rho} F_\rho{}^\nu \right) \\
&= -\frac{1}{8\pi} F^{\rho\sigma} \partial^\nu F_{\rho\sigma} - \frac{1}{4\pi} F^{\mu\rho} \partial_\mu F_\rho{}^\nu - \frac{1}{4\pi} \partial_\mu F^{\mu\rho} F_\rho{}^\nu.
\end{aligned} \tag{4.31}$$

We use the Bianchi identity for the first term and the equation of motion

$$\begin{aligned}
K^\nu &= -\frac{1}{8\pi}F^{\rho\sigma}(\underbrace{-\partial_\rho F_\sigma{}^\nu - \partial_\sigma F^\nu{}_\rho}_{\text{Bianchi}} - 2\partial_\sigma F_\rho{}^\nu) - \frac{1}{c}j^\rho F_\rho{}^\nu \\
&= -\frac{1}{8\pi}\underbrace{F^{\rho\sigma}(\partial_\rho F^\nu{}_\sigma + \partial_\sigma F^\nu{}_\rho)}_{=0} + \frac{1}{c}j^\rho F^\nu{}_\rho \\
&= \rho F^\nu{}_0 + \frac{1}{c}j^k F^\nu{}_k \\
&= \rho F^{\nu 0} - \frac{1}{c}j^k F^{\nu k}. \qquad (4.32)
\end{aligned}$$

Since

$$\begin{aligned}
-j^k F^{0k} &= \mathbf{jE} \\
\rho F^{\ell 0} &= \rho E^\ell \\
j^k F^{\ell k} &= j^k \epsilon^{\ell k m} H^m \qquad (4.33)
\end{aligned}$$

we have the source of the energy-momentum of the EM field

$$K^\mu = (K^0, \mathbf{K}) = \left(\frac{1}{c}\mathbf{jE}, \rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{H}\right). \qquad (4.34)$$

The time-like component is indeed the work done on the charges by the EM field. The spatial components is the rate of change of the momentum of the charges, the Lorentz force.

The energy-momentum density of the EM field $P^\nu = T^{0\nu}$ is

$$\begin{aligned}
P^0 &= \frac{1}{8\pi}(-\mathbf{E}^2 + \mathbf{H}^2) + \frac{1}{4\pi}\mathbf{E}^2 = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2) \\
P^\ell &= \frac{1}{4\pi}F^{0k}F_k{}^\ell = \frac{1}{4\pi}E^k\epsilon^{k\ell m}H^m = -\frac{1}{c}S^\ell \qquad (4.35)
\end{aligned}$$

where the energy flux-density

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} \qquad (4.36)$$

is given by the Poynting vector. In fact, the symmetry of the energy-momentum tensor allows us to identify the energy flux-density with c times the momentum density.

4.4 Electromagnetic waves in the vacuum

Let us consider first the EM field waves in the absence of charges, the solution of the Maxwell equations, (4.15) for $j = 0$. We shall use the Lorentz gauge $\partial_\mu A^\mu = 0$ where the equations of motion are

$$0 = \partial_\nu F^{\mu\nu} = \partial_\nu \partial^\mu A^\nu - \Box A^\mu = -\Box A^\mu. \tag{4.37}$$

we shall consider plane and spherical waves, solutions which display the same value on parallel planes or concentric spheres.

The plane wave solution depends on the combination

$$t^\pm = t \pm \frac{\mathbf{n}\cdot\mathbf{x}}{c} \tag{4.38}$$

of the space-time coordinates. The linearity of the Maxwell equation allows us to write the solution as the linear superposition

$$A_\mu(x) = A^+_\mu(t^+) + A^-_\mu(t^-) \tag{4.39}$$

where

$$\left(\frac{1}{c^2}\partial_t^2 - \Delta\right) A^\pm_\mu(x) = \Box A^\pm_\mu(t^\pm) = 0 \tag{4.40}$$

for arbitrary functions $A^\pm_\mu(t)$, to be determined by the boundary conditions.

The plane waves read in the three-dimensional notation as

$$\begin{aligned}
\mathbf{H} &= \nabla \times \mathbf{A}(t^\pm) = \nabla\left(t \pm \frac{\mathbf{n}\cdot\mathbf{x}}{c}\right) \times \mathbf{A}' = \pm\frac{1}{c}\mathbf{n}\times\mathbf{A}' \\
\mathbf{E} &= -\frac{1}{c}\partial_t \mathbf{A}(t^\pm) - \nabla\phi(t^\pm) = -\frac{1}{c}\mathbf{A}' \mp \frac{1}{c}\mathbf{n}\phi'.
\end{aligned} \tag{4.41}$$

The relation

$$\mathbf{H} = \pm\frac{1}{c}\mathbf{n}\times(-c\mathbf{E} \mp \mathbf{n}\phi') = \mp\mathbf{n}\times\mathbf{E} \tag{4.42}$$

shows that $\mathbf{H}$ orthogonal both to the direction of the propagation, $\mathbf{n}$ and to $\mathbf{E}$. The Lorentz gauge condition,

$$0 = \frac{1}{c}\partial_t\phi + \nabla\cdot\mathbf{A} = \frac{1}{c}\phi' \pm \frac{1}{c}\mathbf{n}\cdot\mathbf{A}' \tag{4.43}$$

together with the second equation in (4.41) shows that $\mathbf{E}$ is orthogonal to $\mathbf{n}$, as well. The energy-momentum density

$$P^\nu = \left(\frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}, -\frac{\mathbf{E}\times\mathbf{H}}{4\pi}\right) = \left(\frac{\mathbf{E}^2}{4\pi}, \pm\frac{\mathbf{E}\times(\mathbf{n}\times\mathbf{E})}{4\pi}\right) = \frac{\mathbf{E}^2}{4\pi}(1, \pm\mathbf{n}), \tag{4.44}$$

is a light-like vector, $P^2 = 0$.

The spherical waves are of the form (4.39) with

$$t^\pm = t \pm \frac{r}{c} \tag{4.45}$$

in spherical coordinate system. We consider them in d spatial dimensions where they satisfy the wave equation $\partial_\mu \partial^\mu A = 0$. We write $A^\pm(x) = r^{\frac{1-d}{2}} a^\pm(t^\pm)$ where $a^\pm$ is a solution of the equation

$$\begin{aligned} 0 &= \left(\frac{1}{c^2}\partial_t^2 - \frac{1}{r^{d-1}}\partial_r r^{d-1}\partial_r\right) A^\pm(t^\pm) \\ &= \left(\frac{1}{c^2}\partial_t^2 + \frac{(d-1)(d-3)}{4r^2}\partial_r - \partial_r^2\right) a^\pm(t^\pm). \end{aligned} \tag{4.46}$$

The functions $a^\pm(t^\pm)$ correspond to 1+1 dimensional plane waves in $d = 1, 3$ only.

Chapter 5

Green functions

The Green functions provide a clear and compact solution of linear equations of motion. But the transparency pf the result hides a drawback, the suppression of the the boundary conditions which are imposed both in space and time. The spatial boundary conditions are usually simpler, they amount to some suppression of the fields at spatial infinity when localized phenomena are investigated. The boundary conditions in time are more complicated and are dealt with briefly in the next section.

5.1 Time arrow problem

The basic equations of Physics, except weak interactions, are invariant under a discrete space-time symmetry, the reversal of the direction of time, $T : (t, \mathbf{x}) \to (-t, \mathbf{x})$. Despite this symmetry, it is a daily experience that the this symmetry is not respected in the world around us. It is enough to recall that we are first born and die later, never in the opposite order. A more tangible example is that the radio transmission arrives at our receivers after its emission, namely the electromagnetic signals travel forward in time rather than backward which is in principle always possible with time reversal invariant equations of motion. What eliminates the backward moving electromagnetic waves? This is one aspect of the time arrow problem in Physics, the problem of pinning down the direction of time, the dynamical origin of the apparent breakdown of the time reversal invariance.

This problem can be discussed at four different level. The most obvious is the level of electromagnetic radiation where it appears as the suppression of backward moving electromagnetic waves in time. It is believed that the origin of this problem is not in Electrodynamics and this property of the electromagnetic waves is related to the boundary conditions chosen in time. We can prescribe the solution we seek in terms of initial or final conditions or even by a mixture of these two possibilities and depending on our choice we see forward or backward going waves or even their mixture in the solution. Why are we interested mainly initial problems rather than final condition problems in physics?

A tentative answer comes from Thermodynamics, the non-decreasing nature of entropy in time. It seems that the composite systems tend to become more complicated and to expand into more irregular regions in the phase space as the time elapses. This property is might not be related to the breakdown of the time reversal invariance because it must obviously hold for either choice of the time arrow. It seems more to have something to do with the nature of the initial conditions we encounter in Physics.

The choice of the initial condition leads us to the astrophysical origin of the time arrow. The current cosmological models, solutions of the formally time reversal invariant Einstein equations of General Relativity, suggest that our Universe undergone a singularity in the distant past. This singular initial condition might be the origin of the peculiar features of the choice of the time arrow.

Yet another level of this issue is the quantum-classical crossover, the scale regime where quantum effects give rise to classical physics. Each measurement traverses this crossover, it magnifies some microscopic quantum effects into macroscopic, classical one. This magnification process, such as the condensation of the drops in the Wilson cloud chamber or the "click" of a Geiger counter indicating th presence of an energetic particle, breaks the time reversal invariance. In fact, the end result of the measurements, a classical "record" created endures the flow of time and can not be reconverted into microscopic phenomena without macroscopic trace. Hence the deepest level of the breakdown of the time reversal invariance comes from the scale regions because any quantum gravitational problem must be handled by this scheme.

Instead of following a more detailed analysis of this dynamical issue we confine the discussion of the separation of the kinematical aspects of this problem. The question we turn to is the way a certain initial of final condition problem can be handled within the framework of Classical Field Theory. The problem arises from the use of the variational principle in deriving the equations of motion. The variational equations of motion can not break the time reversal invariance and can not handle any boundary conditions which does it.

We start the discussion with the formal introduction of the Green function. Let us consider a given function of the time $f(t)$ and the inhomogeneous linear differential equation

$$Lf = g, \tag{5.1}$$

where L is a differential operator acting on the time variable. The Green function is the inverse of the operator L and satisfies the equation

$$L_t G(t, t') = \delta(t - t'). \tag{5.2}$$

The index in L_t is a reminder that the differential operator acts on the variable t of the two variable function $G(t, t')$. Note that for translation invariant L we have $G(t, t') = G(t - t')$. The Dirac-delta is the identify operator on the function space, thus $G = L^{-1}$

The solution of Eq. (5.1) can now formally be written as

$$f(t) = \int dt' G(t, t') g(t'). \tag{5.3}$$

The time reversal invariance of the propagation of perturbations requires that the Green function be symmetric with respect to the exchange of its time variables, $G(t, t') = G(t', t)$. When the propagation of a signal violates time reversal invariance then the Green function must contain antisymmetric part.

The variation principle which reproduces Eq. (5.1) as an equation of motion is based on the action

$$S[x] = \frac{1}{2} \int dt dt' f(t) G^{-1}(t, t') f(t') - \int dt f(t) g(t). \tag{5.4}$$

But the quadratic action is invariant under the exchange of the integral variables $t \leftrightarrow t'$. Therefore, any time reversal breaking antisymmetric part of $G^{-1}(t, t')$ is canceled in the action, the variation principle can not produce time reversal breaking.

The way out of this deadlock is the observation that Eq. (5.2) yields a well defined Green function when the operator L has trivial null-space only. The null-space of an operator is the linear subspace of its domain of definition which is mapped into 0. Whenever there is a non-trivial solution of the equation $Lh = 0$ it can freely be added to the solution of Eq. (5.1), rendering G ill-defined in Eqs. (5.2)-(5.3). The variational problem has nothing to say on the trajectories, corresponding to the null-space of the equation of motion. But this null-space consists of the physically most important functions, the solution of the free equation of motion, in the absence of external source g. This component of the solution must be fixed by the boundary conditions. We shall bring it into the dynamics and the variational equations by adding an infinitesimal, imaginary piece to the inverse propagator,

$$G^{-1} \to G^{-1} + i\mathcal{O}(\epsilon). \tag{5.5}$$

It renders the Green function well defined by making the null-space of G^{-1} trivial and breaks the time reversal invariance in the desired manner because the time reversal implies complex conjugation.

The relation between the time arrow problem and this formal discussion is that these freely addable solutions are to assure the particular boundary conditions. Therefore, the handling of the boundary conditions must come from devices beyond the variational principle, such as the non-symmetrical part of the Green function.

5.2 Invertible linear equation

We start with the simple case where L is invertible and has trivial null-space. The invertible differential operators usually arise in time independent problems. We consider here the case of a static, 3 dimensional equation

$$\Delta f = g \tag{5.6}$$

in the three-volume V when f and $\nabla f_\perp$ are given on ∂V. The null-space of the operator Δ is nontrivial, it consists of harmonic functions. But by imposing boundedness on the solution on an infinitely large domain, a rather usual condition in typical physical cases, the null-space becomes trivial.

One can split the solution as $f = f_{\text{part}} + f_{\text{hom}}$ where f_{part} is a particular solution of the inhomogeneous equation and f_{hom}, the solution of the homogeneous equation. Due to boundedness f_{hom} must be a trivial constant and will be ignored. A useful particular solution is found by inspecting the first two derivatives of the function

$$D(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{5.7}$$

which read as

$$\begin{aligned} \partial_k \frac{1}{|\mathbf{x}|} &= -\frac{x^k}{|\mathbf{x}|^3} \\ \partial_\ell \partial_k \frac{1}{|\mathbf{x}|} &= -\frac{\delta^{k\ell}}{|\mathbf{x}|^3} + 3\frac{x^k x^\ell}{|\mathbf{x}|^5} \end{aligned} \tag{5.8}$$

give

$$\Delta \frac{1}{|\mathbf{x}|} = 0 \tag{5.9}$$

for $\mathbf{x} \neq 0$. Apparently $\Delta \frac{1}{|\mathbf{x}|}$ is a distribution what can be identified by calculating the integral

$$\int_{\mathbf{x}^2<\epsilon^2} dV f(\mathbf{x}) \Delta \frac{1}{|\mathbf{x}|} = -\underbrace{\int_{\mathbf{x}^2<\epsilon^2} dV \nabla f(\mathbf{x}) \cdot \nabla \frac{1}{|\mathbf{x}|}}_{\mathcal{O}(\epsilon)} + \underbrace{\int_{\mathbf{x}^2=\epsilon^2} d\mathbf{S} f(\mathbf{x}) \cdot \nabla \frac{1}{|\mathbf{x}|}}_{-4\pi f(0)} \tag{5.10}$$

giving

$$\Delta_x D(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{5.11}$$

or

$$D(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} | \frac{1}{\Delta} | \mathbf{y} \rangle. \tag{5.12}$$

Thus we have

$$f_{\text{part}}(\mathbf{x}) = \int d^3 y D(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) = -\int d^3 y \frac{g(\mathbf{y})}{4\pi |\mathbf{x} - \mathbf{y}|}. \tag{5.13}$$

To find the homogeneous solution we start with Gauss integral theorem,

$$\int_{\partial V} d\mathbf{S}_y \mathbf{F}(\mathbf{y}) = \int_V d^3 y \nabla \mathbf{F}(\mathbf{y}) \tag{5.14}$$

and by applying for $\mathbf{F}(\mathbf{y}) = D(\mathbf{x}, \mathbf{y}) \nabla f(\mathbf{y}) - f(\mathbf{y}) \nabla D(\mathbf{x}, \mathbf{y})$ we arrive at Green theorem

$$\begin{aligned} &\int_{\partial V} d\mathbf{S}_y [D(\mathbf{x}, \mathbf{y}) \nabla f(\mathbf{y}) - f(\mathbf{y}) \nabla D(\mathbf{x}, \mathbf{y})] \\ &= \int_V d^3 y [D(\mathbf{x}, \mathbf{y}) \Delta f(\mathbf{y}) - f(\mathbf{y}) \Delta_y D(\mathbf{x}, \mathbf{y})]. \end{aligned} \tag{5.15}$$

which gives

$$f(\mathbf{x}) = -\frac{1}{4\pi} \int_V d^3 y \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} + \frac{1}{4\pi} \int_{\partial V} d\mathbf{S}_y \left(\frac{\nabla f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} - f(\mathbf{y}) \nabla_y \frac{1}{|\mathbf{x} - \mathbf{y}|} \right). \tag{5.16}$$

5.3 Non-invertible linear equation with boundary conditions

The non-invertible operators usually appears in dynamical problems. Let us consider the equation

$$\Box f = g \tag{5.17}$$

on 4-dimensional space-time where the function f is sought for a given g. We follow first the extension of the previous argument for four-dimensions: We define a Green-function which is the solution of the equation

$$\Box_x D(x,y) = \delta(x-y) \tag{5.18}$$

and take the time integral of Eq. (5.15) with a translation invariant Green function $D(x,y) = D(x-y)$,

$$\begin{aligned}
&\int_{[t_i,t_f]\otimes\partial V} dtd\mathbf{S}[D(x,y)\nabla f(y) - f(y)\nabla_y D(x,y)] \qquad (5.19)\\
&= \int_{[t_i,t_f]\otimes V} dy[D(x,y)\Delta f(y) - f(y)\Delta_y D(x,y)]\\
&= \int_{[t_i,t_f]\otimes V} dy[-D(x,y)\Box f(y) + D(x,y)\partial_t^2 f(y) - f(y)\partial_{t_y}^2 D(x,y)\\
&+f(y)\Box_y D(x,y)]\\
&= f(x) - \int_{[t_i,t_f]\otimes V} dyD(x,y)\Box f(y)\\
&+ \int_{[t_i,t_f]\otimes V} dy\partial_{t_y}[D(x,y)\partial_t f(y) - f(y)\partial_{t_y} D(x,y)].
\end{aligned}$$

The resulting equation

$$\begin{aligned}
f(x) &= \int_{[t_i,t_f]\otimes V} dyD(x,y)g(y)\\
&+ \int_{[t_i,t_f]\otimes\partial V} dtd\mathbf{S}[D(x,y)\nabla f(y) - f(y)\nabla_y D(x,y)]\\
&- \int_V d^3y[D(x,y)\partial_t f(y) - f(y)\partial_{t_y} D(x,y)]\Big|_{t_i}^{t_f} \qquad (5.20)
\end{aligned}$$

expresses the solution in terms of the boundary conditions, the value of the function f and its derivatives on the boundary of the space-time region where the equation (5.17) is to be solved.

5.4 Retarded and advanced solutions

The definition (5.18) determines the Green-function up to a null-space function, a solution of the homogeneous equation. It is easy to see that the solution (5.20) is well defined and is free of ambiguity. We turn now the more formal method to make the Green-function well defined by introducing an infinitesimal imaginary part. To see better the role of the boundary conditions in time let us drop the spatial boundary conditions by extending the three-volume where the solution of Eq. (5.17) is sought to infinity. The Fourier representation of the Green-function is

$$\tilde{D}(k) = -\frac{1}{k^2} \tag{5.21}$$

for $k^2 \neq 0$ because

$$\int \frac{d^4k}{(2\pi)^4}(-k^2)e^{-ik_\mu x^\mu}\tilde{D}(k) = \int \frac{d^4k}{(2\pi)^4}e^{-ik_\mu x^\mu}. \tag{5.22}$$

To make this integral well defined we have to avoid the singularities of $\tilde{D}(k^2)$ by some infinitesimal shift of the singularities in the complex frequency plane, $k^0 \to k^0 \pm i\epsilon$. The different modifications of the propagator in the vicinity of $k^2 = 0$ introduce different additive homogeneous solutions of Eq. (5.17) in the Green-function.

Let us introduce first the retarded Green-function, $D^r(x,y) \approx \Theta(x^0 - y^0)$ which is used when the initial conditions are known. It is obtained by shifting the poles of $\tilde{D}(k^2)$ slightly below the real axes on the complex energy plane. In fact, the frequency integral

$$D(\mathbf{k},t) = \int \frac{dk^0}{2\pi}e^{-ik^0 t}\tilde{D}(k) \tag{5.23}$$

is non-vanishing just for $t > 0$. The advanced Green-function is used when the final conditions are known and it is obtained by shifting the poles slightly above the real axis,

$$D^{\overset{r}{a}}(x) = -\int \frac{d^3k}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{e^{-ik_\mu x^\mu}}{(k_0 + |\mathbf{k}| \pm i\epsilon)(k_0 - |\mathbf{k}| \pm i\epsilon)} \tag{5.24}$$

The explicit calculation gives

$$
\begin{aligned}
D^r(x) &= -\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{kx}} \int \frac{dk_0}{2\pi} \frac{e^{-ick_0 t}}{(k_0 + i\epsilon - |\mathbf{k}|)(k_0 + i\epsilon + |\mathbf{k}|)} \\
&= i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{kx}} \left(\frac{e^{-ickt}}{2k} - \frac{e^{ickt}}{2k} \right) \\
&= \frac{i}{(2\pi)^3} \int dk k^2 d\phi d(\cos\theta) e^{ikr\cos\theta} \frac{e^{-ickt} - e^{ickt}}{2k} \\
&= \frac{i}{(2\pi)^2} \int dk k^2 \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{e^{-ickt} - e^{ickt}}{2k} \\
&= \frac{1}{2(2\pi)^2 r} \int_0^\infty dk (e^{ikr} - e^{-ikr})(e^{-ickt} - e^{ickt}) \\
&= \frac{1}{8\pi r} \int_{-\infty}^{\infty} \frac{dk}{2\pi} (e^{ik(r-ct)} + e^{ik(-r+ct)} - e^{-ik(r+ct)} - e^{ik(r+ct)}) \\
&= \frac{1}{4\pi r} [\delta(-r+ct) - \delta(r+ct)] \quad (t > 0) \\
&= \frac{\delta(ct - r)}{4\pi r} \quad (t > 0)
\end{aligned}
\tag{5.25}
$$

and

$$
\begin{aligned}
D^a(x) &= -\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{kx}} \int \frac{dk_0}{2\pi} \frac{e^{-ick_0 t}}{(k_0 - i\epsilon - |\mathbf{k}|)(k_0 - i\epsilon + |\mathbf{k}|)} \\
&= i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{kx}} \left(\frac{e^{ickt}}{2k} - \frac{e^{-ickt}}{2k} \right) \\
&= \frac{\delta(r+ct) - \delta(-r+ct)}{4\pi r} \quad (t < 0) \\
&= \frac{\delta(ct + r)}{4\pi r}.
\end{aligned}
\tag{5.26}
$$

Finally we have

$$
\begin{aligned}
f(x) &= \int d^4y D^{\substack{r \\ a}}(x-y) g(y) + f_{\substack{in \\ out}}(x) \\
&= \int d^4y \frac{\delta(x^0 - y^0 \mp |\mathbf{x} - \mathbf{y}|) g(y)}{4\pi |\mathbf{x} - \mathbf{y}|} + f_{\substack{in \\ out}}(x) \\
&= \int d^3y \frac{g\left(t_x \mp \frac{|\mathbf{x}-\mathbf{y}|}{c}, \mathbf{y}\right)}{4\pi |\mathbf{x} - \mathbf{y}|} + f_{\substack{in \\ out}}(x)
\end{aligned}
\tag{5.27}
$$

where $f(t_{\substack{i\\f}}, \mathbf{x}) = f_{\substack{in\\out}}(t_{\substack{i\\f}}, \mathbf{x})$ and $\Box f_{\substack{in\\out}} = 0$. It is easy to find the relativistically invariant form of the Green functions,

$$\begin{aligned} D^{\substack{r\\a}}(x) &= \Theta(\pm t)\frac{\delta(ct \mp r)}{4\pi r} \\ &= \Theta(\pm t)\frac{\delta(ct + r) + \delta(ct - r)}{4\pi r} \\ &= \Theta(\pm t)\frac{\delta(c^2t^2 - r^2)}{2\pi} \\ &= \Theta(\pm x^0)\frac{\delta(x^2)}{2\pi}. \end{aligned} \tag{5.28}$$

There is no dynamical issue in choosing one or other solution. The trivial guiding principle in selecting a Green-function is the information we possess, the initial the final conditions. One can imagine another, rather unrealistic problem where the sum $ag_{in}(x) + (1 - a)g_{out}(x)$ is known to the solution of Eq. (5.17). Then the solution is

$$f(x) = \int dy[aD^r(x, y) + (1 - a)D^a(x, y)]g(y) + ag_{in}(x) + (1 - a)g_{out}(x). \tag{5.29}$$

We are accustomed to think in terms of initial rather than final conditions and therefore use the retarded solutions. This is due to the experimental fact that the homogeneous solution of the Maxwell-equation, the incoming radiation field is negligible compared to the final, outgoing field after some local manipulation. The deep dynamical question is why is this the case, why is the radiation field rather weak for $t \to \infty$ when the basic equations of motion are invariant with respect to the inversion of the direction of the time.

Since $D^a(x, y) = D^{r\,\mathrm{tr}}(x, y) = D^r(y, x)$ the symmetric and antisymmetric Green-functions

$$D^{\substack{n\\f}} = \frac{1}{2}(D^r \pm D^a) \tag{5.30}$$

give the solutions of the inhomogeneous and homogeneous equation, respectively. The inhomogeneous Green-functions are connected by the relation

$$D^{\substack{r\\a}}(x, y) = 2D^n(x, y)\Theta(\pm(x^0 - y^0)) \tag{5.31}$$

where the near field Green function is

$$D^n(x) = \frac{\delta(x^2)}{4\pi} \tag{5.32}$$

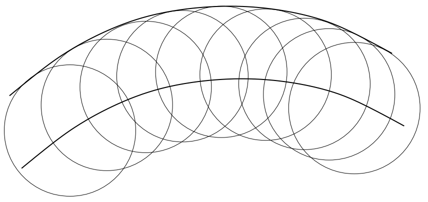

Figure 5.1: Huygens principle for a wave front.

according to Eq. (5.28). The Fourier representation of the homogeneous Green function can be obtained in an obvious manner,

$$\begin{aligned} D^f(x) &= \frac{1}{4\pi}\delta(x^2)\epsilon(x^0) \\ &= -\frac{i}{2}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{kx}}\left(\frac{e^{ickt}}{2k}-\frac{e^{-ickt}}{2k}\right) \\ &= \frac{i}{2}\int\frac{d^4k}{(2\pi)^3}e^{-ikx}\frac{\delta(k_0-|\mathbf{k}|)-\delta(k_0+|\mathbf{k}|)}{2|\mathbf{k}|} \\ &= \frac{i}{2}\int\frac{d^4k}{(2\pi)^3}e^{ikx}\delta(k^2)\epsilon(k_0) \end{aligned} \tag{5.33}$$

where $\epsilon(x) = \mathrm{sign(x)}$. A useful relation satisfied by this Green-function is

$$\begin{aligned} \partial_{x^0}D(x)_{x^0=0} &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{kx}}k_0\frac{\delta(k_0-|\mathbf{k}|)-\delta(k_0+|\mathbf{k}|)}{2|\mathbf{k}|} \\ &= \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{kx}}\frac{1}{2}[\delta(k_0-|\mathbf{k}|)+\delta(k_0+|\mathbf{k}|)] \\ &= \frac{1}{2}\delta(\mathbf{x}) \end{aligned} \tag{5.34}$$

The far field, given by D^f is closely related to the radiation field. The expressions

$$A^{\overset{r}{a}}(x) = \frac{4\pi}{c}\int dyD^{\overset{r}{a}}(x,y)j(y) + A_{\substack{in\\out}}(x) \tag{5.35}$$

suggest the definition

$$A_{rad} = A_{out} - A_{in} = 2A^f. \tag{5.36}$$

Let us close this discussion with a remark about the Huygens principle stating that the wavefront of a propagating light coincide with the envelope of spherical waves emitted by the points of the wavefront at an earlier time. This implies a fixed propagation speed. The retarded Green-function for d space-time dimensions

$$D^r(x) = \begin{cases} \frac{1}{2\pi^{d/2-1}}\Theta(x^0)(\frac{d}{dx^2})^{(d/2-2)}\delta(x^2) & v = c, d \text{ even}, \\ (-1)^{\frac{d-3}{2}}\frac{1}{2\pi^{2/d}}\Gamma(\frac{d}{2}-1)\Theta(x^0-|\mathbf{x}|)(x^2)^{1-d/2} & v \leq c, d \text{ odd}. \end{cases} \tag{5.37}$$

shows that the propagation of the EM wave is restricted to the future light cone in even dimensional space-times only. For odd space-time dimensions the speed of the propagation is not fixed, special relativity takes a radically different form and the Huygens principle is violated.

Chapter 6

Radiation of a point charge

We consider in this chapter a point charge following a prescribed world line and and determine the induced electromagnetic field.

6.1 Liénard-Wiechert potential

As the first step we seek the electromagnetic field $A^\mu(x)$ at $x = (t, \mathbf{x})$ created by a point charge e following the world line $x^\mu(s)$. The current is

$$j^\mu(x) = ce \int ds \delta(x - x(s)) \dot{x}^\mu(s) \tag{6.1}$$

and $A^\mu_{in} = 0$. It is easy to check that at any point x we have a single, definite event on the world line which contribute either to the retarded or to the advanced radiation. In fact, the world-line, having time-like tangent vector can traverse of the future or the past light cone of any point at a single event only as shown in Fig. 6.1. We shall find the answer in two different ways, by a simple heuristic argument and by a more general and complicated manner.

Heuristic method: The charge at $x' = (t', \mathbf{x}')$ can contribute to this field if the difference $x - x'$ is a null-vector,

$$ct - ct' = \pm|\mathbf{x} - \mathbf{x}'| \tag{6.2}$$

(+:retarded propagation, $-$:advanced propagation). In the coordinate system where the charge is at rest at the emission of the electromagnetic field we have

$$\phi = \frac{e}{|\mathbf{x} - \mathbf{x}'|}, \qquad \mathbf{A} = 0. \tag{6.3}$$

Let us generalize this expression for an arbitrary inertial system, in particular where the four vector of the charge in the retarded or advanced time is

$$\dot{x}^\mu = \frac{dx^\mu(s)}{ds} = (c, \mathbf{v})\frac{dt}{ds} \tag{6.4}$$

For this end we introduce the four-vector $R^\mu = (ct - ct', \mathbf{x} - \mathbf{x}')$ and write

$$A^{\overset{r}{a}\mu} = \pm\frac{e\dot{x}^\mu}{R\cdot\dot{x}} \tag{6.5}$$

Due to $R\cdot\dot{x} = (rc - \mathbf{r}\mathbf{v})\frac{dt}{ds}$, $r = |\mathbf{x} - \mathbf{x}'|$ we have

$$\phi = \pm\frac{e}{r - \frac{\mathbf{r}\cdot\mathbf{v}}{c}}, \quad \mathbf{A} = \pm\frac{e\mathbf{v}}{c(r - \frac{\mathbf{r}\cdot\mathbf{v}}{c})}. \tag{6.6}$$

The part $\mathcal{O}\left(\mathbf{v}^0\right)$ is the static Coulomb potential, the $\mathbf{v}$-dependent pieces in the denominator represent the retardation or advanced effects. Finally, $\mathbf{A}$ gives the magnetic field induced from the Coulomb potential by the Lorentz boost.

The more systematical way of obtaining the induced field is based on the use of the Green-functions,

$$\begin{aligned} A^{\overset{r}{a}\mu}(x) &= e4\pi\int dx'\int dsD^{\overset{r}{a}}(x-x')\delta(x'-x(s))\dot{x}^\mu(s) \\ &= 2e\int dx'\int ds\delta((x-x')^2)\Theta(\pm(x^0-x'^0))\delta(x'-x(s))\dot{x}^\mu(s) \\ &= 2e\int ds\delta((x-x(s))^2)\Theta(\pm(x^0-x^0(s)))\dot{x}^\mu(s) \end{aligned} \tag{6.7}$$

$x - x(s^{\overset{r}{a}})$ can be written as the linear superposition of two orthogonal vectors,

$$x - x(s^{\overset{r}{a}}) = (\pm\dot{x} + w)R^{\overset{r}{a}} \tag{6.8}$$

where w is space-like. Since $(x - x(s^{\overset{r}{a}}))^2 = 0$, $\dot{x}^2 = 1$ and $\dot{x}\cdot w = 0$ we have $w^2 = -1$ and

$$R^{\overset{r}{a}} = -w\cdot(x - x(s^{\overset{r}{a}})) = \pm\dot{x}\cdot(x - x(s^{\overset{r}{a}})) \tag{6.9}$$

The use of the rule $\delta(f(x)) \to \delta(x - x_0)/|f'(x_0)|$ where $f(x_0) = 0$ and the relation

$$\frac{d(x - x(s))^2}{ds} = \mp 2R^{\overset{r}{a}} \tag{6.10}$$

gives

$$A^{\overset{r}{a}\mu}(x) = e\frac{\dot{x}^\mu(s^{\overset{r}{a}})}{R^{\overset{r}{a}}}. \tag{6.11}$$

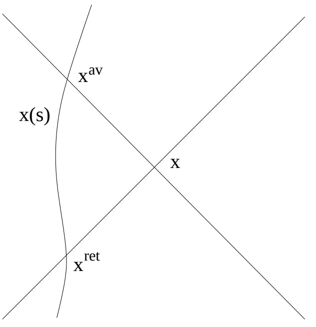

Figure 6.1: The observer at x receives the signal emitted from the point x^{ret} or x^{av} for the retarded or the advanced propagation, respectively.

6.2 Field strengths

The field strength is obtained by calculating the space-time derivatives of the Liénard-Wiechert potential (6.11),

$$
\begin{aligned}
\partial_\mu A^\nu(x) &= e4\pi \int dx' \int ds \partial_{x^\mu} D^r(x-x')\delta(x'-x(s))\dot{x}^\nu(s) \\
&= e4\pi \int ds \frac{\partial D^r(x-x(s))}{\partial (x-x(s))^2} \frac{\partial (x-x(s))^2}{\partial x^\mu} \dot{x}^\nu(s) \\
&= e8\pi \int ds \frac{\partial D^r(x-x(s))}{\partial s} \underbrace{\frac{\partial s}{\partial (x-x(s))^2}}_{1/[-2(x-x(s))\cdot\dot{x}(s)]} (x-x(s))_\mu \dot{x}^\nu(s) \\
&= -e4\pi \int ds \frac{\partial D^r(x-x(s))}{\partial s} \frac{(x-x(s))_\mu \dot{x}^\nu(s)}{(x-x(s))\cdot\dot{x}} \\
&= \underbrace{-e4\pi D^r(x-x(s)) \frac{(x-x(s))_\mu \dot{x}^\nu(s)}{(x-x(s))\cdot\dot{x}} \Big|_{-\infty}^{\infty}}_{=0} \\
&\quad + e4\pi \int ds D^r(x-x(s)) \frac{\partial}{\partial s} \frac{(x-x(s))_\mu \dot{x}^\nu(s)}{(x-x(s))\cdot\dot{x}(s)} \\
&= 2e \int ds \delta((x-x(s))^2)\Theta(x^0-x^0(s)) \frac{\partial}{\partial s} \frac{(x-x(s))_\mu \dot{x}^\nu(s)}{(x-x(s))\cdot\dot{x}(s)} \\
&= e \frac{1}{(x-x(s))\cdot\dot{x}(s)} \frac{\partial}{\partial s} \frac{(x-x(s))_\mu \dot{x}^\nu(s)}{(x-x(s))\cdot\dot{x}(s)}_{|s=s^r}
\end{aligned} \tag{6.12}
$$

The introduction of the scalar

$$Q = (x - x(s)) \cdot \ddot{x}(s) = R(\pm\dot{x} + w) \cdot \ddot{x}(s) = Rw \cdot \ddot{x}^\mu(s) \tag{6.13}$$

allows us to write

$$\begin{aligned}
F^{\mu\nu} &= \frac{e}{R^3}[(x - x(s))^\mu \ddot{x}^\nu R - \dot{x}^\mu \dot{x}^\nu R - (x - x(s))^\mu \dot{x}^\nu Q \\
&\quad + \dot{x} \cdot \dot{x}(x - x(s))^\mu \dot{x}^\nu - (\mu \leftrightarrow \nu)] \\
&= \frac{e}{R^3}[(x - x(s))^\mu \ddot{x}^\nu R - (x - x(s))^\mu \dot{x}^\nu Q + (x - x(s))^\mu \dot{x}^\nu - (\mu \leftrightarrow \nu)] \\
&= \frac{e}{R^2}[(\pm\dot{x} + w)^\mu \ddot{x}^\nu R - (\pm\dot{x} + w)^\mu \dot{x}^\nu Q + (\pm\dot{x} + w)^\mu \dot{x}^\nu - (\mu \leftrightarrow \nu)] \\
&= \frac{e}{R^2}[(\pm\dot{x} + w)^\mu \ddot{x}^\nu R - w^\mu \dot{x}^\nu R w \cdot \ddot{x} + w^\mu \dot{x}^\nu - (\mu \leftrightarrow \nu)]
\end{aligned} \tag{6.14}$$

The field strength is the sum of $\mathcal{O}\left(R^{-1}\right)$ and $\mathcal{O}\left(R^{-2}\right)$ terms, called far and near fields.

The tree-dimensional notation is introduced by

$$\dot{x} = (c, \mathbf{v})\frac{dt}{ds}, \qquad R^{\overset{r}{a}} = \pm(\pm rc - \mathbf{r}\mathbf{v})\frac{dt}{ds} = (rc \mp \mathbf{r}\mathbf{v})\frac{dt}{ds} = \frac{r \mp \mathbf{r}\beta}{\sqrt{1-\beta^2}} \tag{6.15}$$

where $\beta = \frac{\mathbf{v}}{c}$. The formally introduced spatial unit vector w is determined by the condition

$$(\pm r, \mathbf{r}) = R\left[\pm(c, \mathbf{v})\frac{dt}{ds} + w\right] \tag{6.16}$$

which yields

$$\begin{aligned}
w &= \frac{1}{R}(\pm r, \mathbf{r}) \mp (c, \mathbf{v})\frac{dt}{ds} \\
&= \frac{(\pm r, \mathbf{r})}{(rc \mp \mathbf{r}\mathbf{v})}\frac{ds}{dt} \mp (c, \mathbf{v})\frac{dt}{ds}
\end{aligned} \tag{6.17}$$

which reads

$$\begin{aligned}
\mathbf{w} &= \frac{\mathbf{r}}{(r \mp \mathbf{r}\beta)}\sqrt{1-\beta^2} \mp \frac{\beta}{\sqrt{1-\beta^2}} \\
w^0 &= \pm\frac{r}{(rc \mp \mathbf{r}\mathbf{v})}\frac{ds}{dt} \mp c\frac{dt}{ds} \\
&= \pm\left[\frac{r}{(r \mp \mathbf{r}\beta)}\sqrt{1-\beta^2} - \frac{1}{\sqrt{1-\beta^2}}\right]
\end{aligned} \tag{6.18}$$

in three-dimensional notation.

The near-field depends on the coordinate and the velocity,

$$
\begin{aligned}
F^{n\mu\nu} &= \frac{e}{R^3}[(x-x(s))^\mu \dot{x}^\nu - (x-x(s))^\nu \dot{x}^\mu] \\
&= \frac{e}{R^2}[w^\mu \dot{x}^\nu - w^\nu \dot{x}^\mu] \qquad (6.19)
\end{aligned}
$$

The near electric field, $E_j = F^{j0}$, is

$$
\begin{aligned}
\mathbf{E}^n &= \frac{e}{R^2}[\mathbf{w}\dot{x}^0 - w^0 \dot{\mathbf{x}}] \\
&= \frac{e(1-\beta^2)}{(r \mp \mathbf{r}\beta)^2}\left[\left(\frac{\mathbf{r}}{(r \mp \mathbf{r}\beta)}\sqrt{1-\beta^2} \mp \frac{\beta}{\sqrt{1-\beta^2}}\right)\frac{1}{\sqrt{1-\beta^2}}\right. \\
&\quad \left. \mp \left(\frac{r}{(r \mp \mathbf{r}\beta)}\sqrt{1-\beta^2} - \frac{1}{\sqrt{1-\beta^2}}\right)\frac{\beta}{\sqrt{1-\beta^2}}\right] \\
&= \frac{e(1-\beta^2)}{(r \mp \mathbf{r}\beta)^2}\left[\left(\frac{\mathbf{r}}{(r \mp \mathbf{r}\beta)} \mp \frac{\beta}{1-\beta^2}\right) \mp \left(\frac{r}{(r \mp \mathbf{r}\beta)} - \frac{1}{1-\beta^2}\right)\beta\right] \\
&= \frac{e(1-\beta^2)}{(r \mp \mathbf{r}\beta)^2}\left(\frac{\mathbf{r}}{(r \mp \mathbf{r}\beta)} \mp \frac{r\beta}{(r \mp \mathbf{r}\beta)}\right) \\
&= e\frac{(1-\beta^2)(\mathbf{r} \mp r\beta)}{(r \mp \mathbf{r}\cdot\beta)^3} \\
&= e\frac{(1-\frac{v^2}{c^2})(\mathbf{r} \mp r\frac{\mathbf{v}}{c})}{(r \mp \frac{\mathbf{r}\cdot\mathbf{v}}{c})^3} \qquad (6.20)
\end{aligned}
$$

The far-field depends on the acceleration as well,

$$
F^{f\mu\nu} = \frac{e}{R^3}[(x-x(s))^\mu(\ddot{x}^\nu R - \dot{x}^\nu Q) - (x-x(s))^\nu(\ddot{x}^\mu R - \dot{x}^\mu Q)] \qquad (6.21)
$$

and

$$
\mathbf{E}^f = e\frac{\mathbf{r}\times[(\mathbf{r} - r\frac{\mathbf{v}}{c})\times\mathbf{a}]}{c^2(r - \frac{\mathbf{r}\cdot\mathbf{v}}{c})^3}, \qquad (6.22)
$$

where $\mathbf{a} = \frac{d\mathbf{v}}{dt}$. We have the relation

$$
\mathbf{H} = \frac{\mathbf{r}\times\mathbf{E}}{r} \qquad (6.23)
$$

for both fields.

6.3 Dipole radiation

The complications in obtaining the Liénard-Wiechert potential come from the retardation. Thus it is advised to see the limits when the retardation effects are weak and the final result can be expanded in them. Let us suppose that the characteristic time and distance scales of the prescribed charge distribution are t_{ch} and r_{ch}, respectively. The period length of the radiation is approximately t_{ch}, yielding the wavelength $\lambda = ct_{ch}$. The retardation time is when is needed for the EM wave to traverse the charge distribution, $t_{ret} = r_{ch}/c$. The retardation effects are therefore weak for $t_{ret}/t_{ch} \ll 1$ which gives $r_{ch} \ll \lambda$. Another way to express this inequality is to consider the characteristic speed of the charge system, $v_{ch} = r_{ch}/t_{ch}$, to write $\lambda = cr_{ch}/v_{ch}$ which yields $v_{ch} \ll c$.

We assume that these inequalities hold and consider the leading order effect of the retardation on the retarded Liénard-Wiechert potential (6.11)

$$A(x) = \sum_a \frac{e_a \dot{x}^\mu(s_a^r)}{R_a^r}. \tag{6.24}$$

It is sufficient to find the magnetic field,

$$\mathbf{A}(x) = \frac{1}{cr}\sum_a e_a \mathbf{v}_a \left(1 + \mathcal{O}\left(\frac{|\mathbf{v}|}{c}\right)\right) \tag{6.25}$$

where r denotes the distance between the observation point and the center of the charges and $\mathbf{v}_a$ stands for the velocity of the charge a at the time of observation. Since

$$\sum_a e_a \mathbf{v}_a = \frac{d}{dt}\sum_a e_a \mathbf{x}_a = \frac{d}{dt}\mathbf{d} \tag{6.26}$$

where $\mathbf{d}$ is the dipole moment of the charge system we have

$$\mathbf{A}(x) = \frac{1}{cr}\frac{d\mathbf{d}}{dt} \tag{6.27}$$

in the leading order. Then the magnetic field is given by

$$\begin{aligned} \mathbf{H} &= \nabla \times \frac{1}{cr}\frac{d\mathbf{d}(t-\frac{r}{c})}{dt} \\ &= -\frac{1}{cr^2}\mathbf{n} \times \frac{d\mathbf{d}}{dt} - \frac{1}{c^2 r}\mathbf{n} \times \frac{d^2\mathbf{d}}{dt^2} \end{aligned} \tag{6.28}$$

which reduces to the far field

$$\mathbf{H} = \frac{1}{c^2 r}\frac{d^2\mathbf{d}}{dt^2} \times \mathbf{n} = \frac{1}{c^2 r}\sum_a e_a \mathbf{a}_a \times \mathbf{n} \tag{6.29}$$

for the retarded solution. Since the vectors $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{n}$ form an orthogonal basis we have

$$\mathbf{E} = \frac{1}{c^2 r}\left(\frac{d^2\mathbf{d}}{dt^2} \times \mathbf{n}\right) \times \mathbf{n} = \frac{1}{c^2 R}\sum_a e_a(\mathbf{a}_a \times \mathbf{n}) \times \mathbf{n} \tag{6.30}$$

The far field, dipole radiation depends on the acceleration of the charges only.

The radiation power passing trough a surface $d\mathbf{f}$ is

$$dI = \mathbf{S} d\mathbf{f} \tag{6.31}$$

where $\mathbf{S}$ is the Poynting vector and is given by

$$dI = \frac{c}{4\pi} H^2 r^2 d\Omega \tag{6.32}$$

according to Eq. (4.44) where $d\Omega$ denotes the solid angle. In the case of the dipole radiation we find

$$dI = \frac{1}{4\pi c^3}\left(\frac{d^2\mathbf{d}}{dt^2} \times \mathbf{n}\right)^2 d\Omega = \frac{1}{4\pi c^3}\left(\frac{d^2\mathbf{d}}{dt^2}\right)^2 \sin^2\theta d\Omega \tag{6.33}$$

where θ is the angle between $\frac{d^2}{dt^2}\mathbf{d}$ and $\mathbf{n}$. The total radiated power is obtained by integrating over the solid angle,

$$I = \int d\phi \int d(\cos\theta) \frac{1}{4\pi c^3}\left(\frac{d^2\mathbf{d}}{dt^2}\right)^2 \sin^2\theta = \frac{2}{3c^3}\left(\frac{d^2\mathbf{d}}{dt^2}\right)^2. \tag{6.34}$$

For a single charge we have

$$I = \frac{2\mathbf{a}^2}{3c^3}. \tag{6.35}$$

(J. Larmor, 1897).

Chapter 7

Radiation back-reaction

The charges and the electromagnetic fields interact in electrodynamics. The full dynamical problem where both the charges and the electromagnetic field are allowed to follow the time dependence described by their dynamics, the mechanical equation of motion and the Maxwell equations is quite a wonderful mathematical problem. A simpler question is when the motion is partially restricted, when one members of this system is forced to follow a prescribed time dependence and the other is allowed to follow its own dynamics only. For instance, the world lines of a point charge moving in the presence of a fixed electromagnetic field can easily be found by integrating the equation of motion (4.4). The use of the Green functions provides the solution for a number or engineering problems in electrodynamics where the electromagnetic fields are sought for a given charge distribution. We devote this chapter to a question whose complexity is in between the full and the restricted dynamical problems but appears a more fundamental issue.

7.1 The problem

Let us consider a charge moving under the influence of a nonvanishing external force. The force accelerates it and in turn radiation is emitted. The radiation has some energy and momentum which is lost in the supposedly infinite space surrounding the charge. Thus the energy and momentum of our charge is changed and we have to assume that there is some additional force acting on the charge.

The very question is rather perplexing because one would have thought that the equation of motion for the charge, Eq. (4.4), containing the Lorentz force, the second term on the right hand side is the last word in this issue.

There is apparently another force in the "true" equation of motion! The complexity of this problem explains that this is perhaps the last open chapter of classical electrodynamics.

There is a further, even more disturbing question. Does a point-like charge interact with the electromagnetic field induced by its own motion? It is better not, otherwise we run into the problem of singularities like a point-charge at rest at the singular point of its own Coulomb-field. But the electric energy of a given static charge distribution $\rho(\mathbf{x})$,

$$E = \frac{1}{2} \int d^3x d^3y \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \tag{7.1}$$

suggests that the answer is affirmative.

The radiation reaction force whose derivation is the goal of this chapter touches a number of subtle issues.

1. The problem about the limit $r_0 \to 0$ where r_0 is the characteristic size of the charge distribution of a particle raises the possibility that the limits $\hbar \to 0$ and $r_0 \to 0$ do not commute. In fact, in discussing a point charge in classical electrodynamics one tacitly takes the limit $\hbar \to 0$ first to get the laws of classical electrodynamics where the limit $r_0 \to 0$ is performed at the end. But a strongly localized particle induces quantum effects what should be taken into account by keeping $\hbar$ finite, ie. we should start with the limit $r_0 \to 0$ to introduce a point-like particle in Quantum Mechanics and the classical limit $\hbar \to 0$, should be performed at the end only. There are no point charges in this scheme because even if one starts with a strictly point-like charge the unavoidable vacuum-polarization effect generates a charge density polarization cloud of the size of the Compton wavelength around the point charge.

2. Is there regular solutions at all for the set of coupled equations for point charges and the electromagnetic field? It may happen that some smearing, provided by the unavoidable vacuum polarization of quantum electrodynamics is needed to render the solution of the classical equations of motion regular.

3. The existence of the radiation back reaction force is beyond doubt but its derivation is non-trivial. It is a friction force, describing the loss of energy to the radiation field, and can not be derived by variational principle ie. it is not present in the usual variational system of equations of motion of classical electrodynamics.

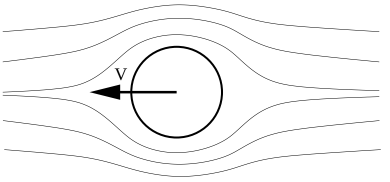

Figure 7.1: A body moving in viscous fluid.

4. The energy radiated out by the charge can not be recovered anymore in an infinite system. Thus the sign of the radiation reaction force represents a dynamical breakdown of the time inversion invariance of the basic laws of electrodynamics.

5. The radiation back reaction force acting on point charges can be calculated exactly and turns out to be proportional to the third derivative of the coordinates. Such kind of force generates self-accelerating motion which is unacceptable.

7.2 Hydrodynamical analogy

Before embarking the detailed study of classical electrodynamics let us consider a simpler, related problem in hydrodynamics, in another classical field theory. We immerse a spherical rigid body of mass M in a viscous fluid as depicted in Fig. 7.1. What is the equation of motion for the center of mass $\mathbf{x}$ of this body?

The naive answer,

$$M\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}_{ext} \tag{7.2}$$

the right hand side being the external force acting on the body is clearly inadequate because it ignores the environment of the body. The full equation of motion must contain a rather involved friction force $\mathbf{F}_{fl}(\mathbf{v})$,

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + \mathbf{F}_{fl}(\mathbf{v}) \tag{7.3}$$

where $\mathbf{v}$ is the velocity of the body.

There are two ways to find the answer. The direct, local one is to construct the force $\mathbf{F}_{ext}(\mathbf{v})$ the fluid exerts on the body by the detailed study of the flow in is vicinity. If we have not enough information to accomplish this calculation then another, more indirect global possibility is to calculate the total momentum $\mathbf{P}_{fl}(\mathbf{v})$ of the fluid which is usually easier to find and to set

$$\mathbf{F}_{fl}(\mathbf{v}) = -\frac{d}{dt}\mathbf{P}_{fl}(\mathbf{v}), \tag{7.4}$$

or equivalently to state that the total momentum of the body is

$$\mathbf{P}_{tot}(\mathbf{v}) = M\mathbf{v} + \mathbf{P}_{fl}(\mathbf{v}). \tag{7.5}$$

We shall find the electromagnetic analogy of both schemes in the rest of this chapter.

7.3 Radiated energy-momentum

We start the establishment of the energy-momentum balance for accelerating charges by considering the energy loss for a slow, non-relativistic charges. In the absence or other intrinsic scales the EM field generated by slow motion agrees with the dipole radiation and the far field expression (6.22)-(6.23) gives the total radiation power (6.35), indicating the presence of forces acting on accelerating charges. We regard now the EM field of a point charge in a more detailed manner.

The field strength of the general, relativistic case satisfies few important relations. The equation

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 0 \tag{7.6}$$

follows from the symmetry of $F_{\mu\nu}F_{\rho\sigma}$ for $\mu \leftrightarrow \sigma$ according to Eq. (6.14). In a similar manner,

$$\tilde{F}^{\rho\sigma}(x - x(s))_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}(x - x(s))_\sigma = 0 \tag{7.7}$$

follows from the symmetry of $F_{\mu\nu}(x - x(s))_\sigma$ for $\nu \leftrightarrow \sigma$. Thus $\mathbf{H} \perp \mathbf{E}$, $\mathbf{H} \perp \mathbf{x} - \mathbf{x}(s)$ and $\mathbf{E} \perp \mathbf{x} - \mathbf{x}(s)$.

The far field satisfies beyond Eqs. (7.6)-(7.7) the conditions

$$F^f_{\mu\nu}F^{f\mu\nu} = 0 \tag{7.8}$$

and

$$F^f_{\mu\nu}(x - x(s))^\nu = 0, \tag{7.9}$$

ie. $|\mathbf{H}^f| = |\mathbf{E}^f|$.

The null-field is defined by the properties $F^{\mu\nu}F_{\mu\nu} = \tilde{F}^{\mu\nu}F_{\mu\nu} = 0$. The far field is null-field, therefore its energy-momentum tensor is

$$\begin{aligned}
T^{f\mu\nu} &= \frac{1}{4\pi}F^{f\mu\sigma}F^{f\nu}_{\sigma} \\
&= \frac{e^2}{4\pi R^6}[(x-x(s))^\mu(\ddot{x}^\sigma R - \dot{x}^\sigma Q) - (x-x(s))^\sigma(\ddot{x}^\mu R - \dot{x}^\mu Q)] \\
&\quad \times[(x-x(s))_\sigma(\ddot{x}^\nu R - \dot{x}^\nu Q) - (x-x(s))^\nu(\ddot{x}_\sigma R - \dot{x}_\sigma Q)]. \quad (7.10)
\end{aligned}$$

Due to $(x-x(s))^2 = 0$ we have

$$\begin{aligned}
T^{f\mu\nu} &= \frac{e^2}{4\pi R^6}[(x-x(s))^\mu(\ddot{x}^\nu R - \dot{x}^\nu Q)(\ddot{x}R - \dot{x}Q)\cdot(x-x(s)) \\
&\quad -(x-x(s))^\mu(x-x(s))^\nu(\ddot{x}R - \dot{x}Q)^2 \\
&\quad +(\ddot{x}^\mu R - \dot{x}^\mu Q)(x-x(s))^\nu(x-x(s))\cdot(\ddot{x}R - \dot{x}Q)]. \quad (7.11)
\end{aligned}$$

The relation

$$\begin{aligned}
(\ddot{x}R - \dot{x}Q)\cdot(x-x(s)) &= [\ddot{x}\dot{x}\cdot(x-x(s)) - \dot{x}\ddot{x}\cdot(x-x(s))]\cdot(x-x(s)) \\
&= [\ddot{x}\cdot(x-x(s)][\dot{x}\cdot(x-x(s))] \\
&\quad -[\dot{x}\cdot(x-x(s))][\ddot{x}\cdot(x-x(s))] = 0
\end{aligned}$$

allows us to write

$$\begin{aligned}
T^{f\mu\nu} &= -\frac{e^2}{4\pi R^6}(x-x(s))^\mu(x-x(s))^\nu(\ddot{x}R - \dot{x}Q)^2 \\
&= -\frac{e^2}{4\pi R^6}(x-x(s))^\mu(x-x(s))^\nu(\ddot{x}^2R^2 + Q^2) \quad (7.12)
\end{aligned}$$

because $\dot{x}\cdot\ddot{x} = 0$.

The radiation reaction four-force acting on the charge, $K^\mu = -\partial_\nu T^{\nu\mu}$, can be obtained by considering the integral I of $\partial_\nu T^{\nu\mu}$ over the four-volume V of Fig. 7.2, bounded by the hyper-surfaces S_1, C_1, S_2, C_2. For sufficiently far from the charges the far field survives only and we have

$$\begin{aligned}
I &= \int dV \partial_\nu T^{\nu\mu} \\
&= \int_{\partial V} dS_\nu T^{\nu\mu} \\
&= \int_{S_2} dS_\nu T^{\nu\mu} - \int_{S_1} dS_\nu T^{\nu\mu} + \int_{C_1} dS_\nu T^{\nu\mu} + \int_{C_2} dS_\nu T^{\nu\mu} \quad (7.13)
\end{aligned}$$

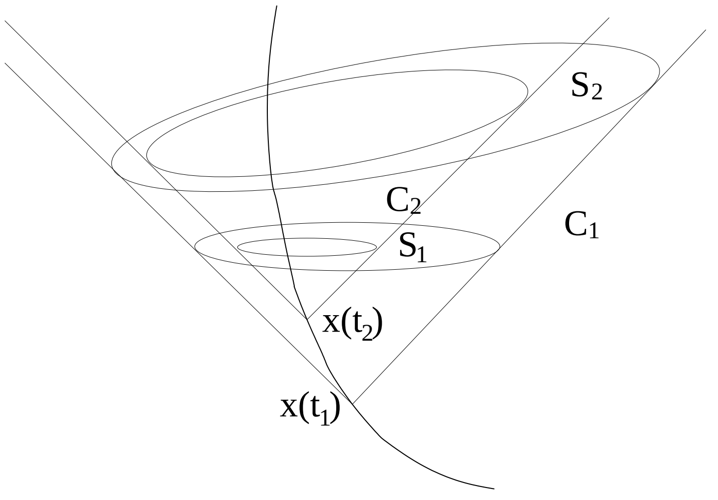

Figure 7.2: Energy momentum emitted by an accelerating charge

for the far field contributions. The last two terms are vanishing because $T^{f\nu\mu} \approx (x - x(s))^\nu$ and $dS_\nu T^{\nu\mu} = 0$ and no energy-momentum crosses the hyper-surfaces C_1 and C_2. It is important to note that this is not true for the near field because $T^{n\mu\nu} \not\approx (x - x(s))^\nu$, the near-field, eg. Coulomb field moves with the charge. The co-moving nature of the near-field contrasted with the decoupled, freely propagating nature of the far field which defines identifies the radiation field.

Since the energy-momentum tensor $t^{\mu\nu}$ of the localized charge is vanishing in the integration volume the energy-momentum conservation $\partial_\nu(T^{\nu\mu} + t^{\nu\mu}) = 0$ assures $I = 0$, and

$$\Delta P^\mu = -\int_S dS_\nu T^{f\nu\mu}, \tag{7.14}$$

the radiated energy-momentum, is a four-vector and is independent of the choice of the surface S.

To calculate ΔP we choose a suitable surface S in such a manner that dS^μ is space-like. We write $x - x(s) = R\dot{x} + y$ where $y \cdot \dot{x} = 0$ and define $dS^\mu = y^\mu R d\Omega ds$. S becomes a sphere of radius R, $y^2 = -R^2$ in the rest-frame of the charge at the emission of the radiation for infinitesimal proper length ds and we have

$$\begin{aligned}\Delta P^\mu &= -ds\int d\Omega R y_\nu T^{f\mu\nu} \\ &= \frac{e^2}{4\pi}ds\int d\Omega y_\nu \frac{(R\dot{x}+y)^\mu(R\dot{x}+y)^\nu}{R^3}\left(\ddot{x}^2+\frac{Q^2}{R^2}\right) \\ &= -\frac{e^2}{4\pi}ds\int d\Omega\frac{(R\dot{x}+y)^\mu}{R}\left(\ddot{x}^2+\frac{Q^2}{R^2}\right). \end{aligned} \tag{7.15}$$

Since

$$\int d\Omega y^\mu = 0 \tag{7.16}$$

for a sphere and

$$Q^2 = [(x-x(s))\cdot\ddot{x}]^2 = [(R\dot{x}-y)\cdot\ddot{x}]^2 = (y\cdot\ddot{x})^2 = y^2\ddot{x}^2\cos^2\theta = -R^2\ddot{x}^2\cos^2\theta \tag{7.17}$$

the energy-flux for relativistic charge is

$$\begin{aligned}\Delta P^\mu &= -\frac{e^2}{4\pi}ds\dot{x}^\mu\int d\Omega\left(\ddot{x}^2+\frac{Q^2}{R^2}\right) \\ &= -\frac{e^2}{4\pi}ds\dot{x}^\mu\ddot{x}^2\int d\Omega(1-\cos^2\theta) \\ &= -\frac{e^2}{2}\dot{x}^\mu\ddot{x}^2 ds\underbrace{\int_{-1}^{1} d(\cos\theta)\sin^2\theta}_{\frac{4}{3}}. \end{aligned} \tag{7.18}$$

and

$$\frac{\Delta P^\mu}{ds} = -\frac{2}{3}e^2\dot{x}^\mu\ddot{x}^2. \tag{7.19}$$

7.4 Brief history

We summarize the stages the radiation reaction force has passed with more attention payed to relatively recent developments.

7.4.1 Extended charge distribution

The energy of a charge e distributed in a sphere of radius r_0 which moves with velocity $\mathbf{v}$, was written by Thomson as $E = K + E_{ed}$ where $K = \frac{1}{2}m_{mech}\mathbf{v}^2$ and

$$E_{em} = \frac{1}{2}\int d^3x(\mathbf{E}^2+\mathbf{H}^2), \tag{7.20}$$

and the actual calculation [5] yields

$$E_{em} = f\frac{e^2}{r_0c^2}\frac{\mathbf{v}^2}{2}, \tag{7.21}$$

f being a dimensionless constant depending on the charge distribution with value $f = 2/3$ for uniformly distributed charge within the sphere. One can introduce the electromagnetic mass for such charge distribution,

$$m_{ed} = \frac{2}{3}\frac{e^2}{r_0c^2} \tag{7.22}$$

giving

$$E_{em} = \frac{m_{ed}}{2}v^2. \tag{7.23}$$

We thereby recover $E = \frac{m}{2}v^2$ where $m = m_{mech} + m_{ed}$. Assuming pure electromagnetic origin of the mass, $m_{mech} = 0$, we have the classical charge radius

$$r_{cl} = \frac{2}{3}\frac{e^2}{mc^2}, \tag{7.24}$$

the distance where the non-mechanical origin of the mass becomes visible.

The next step was made by Lorentz who held the conviction that all electrodynamics phenomena arise from the structure of the electron [6]. Larmor's formule gives

$$\Delta E_L = \frac{2e^2}{3c^3}\int \mathbf{a}^2 dt = \frac{2e^2}{3c^3}\int\left(\frac{d(\mathbf{a}\cdot\mathbf{v})}{dt} - \frac{d\mathbf{a}}{dt}\cdot\mathbf{v}\right)dt \tag{7.25}$$

for the energy loss due to radiation. The contribution of the first term in the last equation is negligible for long time and motion with bounded velocity and acceleration and we have

$$\begin{aligned}\Delta E_L &\approx -\frac{2e^2}{3c^3}\int\frac{d\mathbf{a}}{dt}\cdot\mathbf{v}dt \\ &= -\int \mathbf{F}_{rad}d\mathbf{x}\end{aligned} \tag{7.26}$$

yielding the first time first an expression for the radiation reaction force,

$$\mathbf{F}_L = \frac{2e^2}{3c^3}\frac{d\mathbf{a}}{dt}. \tag{7.27}$$

In another work [7] Lorentz sets out to calculate the direct (Lorentz) force acting on a rigid charge distribution $\rho(\mathbf{x})$ of size r_0 due to the radiation back-reaction. He found

$$\begin{aligned}\mathbf{F}_{rad} &= \rho\left(\mathbf{E}_{rad} + \frac{1}{c}\mathbf{v}\times\mathbf{H}_{rad}\right)\\ &= \underbrace{-\frac{4}{3c^2}\frac{1}{2}\int\frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3xd^3x'}_{\frac{4}{3}m_{ed}}\mathbf{a} + \underbrace{\frac{2e^2}{3c^3}\frac{d\mathbf{a}}{dt}}_{\mathbf{F}_L}\\ &\quad -\frac{2e^2}{3c^3}\sum_{n=2}^{\infty}\frac{(-1)^n}{n!c^n}\frac{d^m\mathbf{a}}{dt^n}\mathcal{O}\left(r_0^{n-1}\right)\end{aligned}$$

The problems, opened by this result are the following.

1. The electromagnetic mass, with its factor 3/2 in front of m_{ed}, given by Eq. (7.22) differs from Thomson's result.
2. Higher order derivatives with respect to the time appear in the equation of motion. They contradict with our daily experience in mechanics.
3. There is no place in electrodynamics for cohesive forces, appearing for finite charge distribution, $r_0 \neq 0$, in equilibrium.
4. The divergence $m_{ed} = \infty$ in the limit $r_0 \to 0$ spoils our ideas about point charges.

While Lorentz concentrated on the energy loss of the charge system Abraham approached the problem from the point of view of the momentum loss. He identified the momentum of the Coulomb field of an charge in uniform motion by its Poyting's vector [8],

$$\mathbf{p}_{rad} = \frac{1}{4\pi c^2}\int(\mathbf{E}\times\mathbf{H})d^3x. \tag{7.28}$$

The actual calculation yields

$$\mathbf{p}_{rad} = \frac{4}{3}m_{ed}\mathbf{v} \tag{7.29}$$

where the electromagnetic mass is given by Eq. (7.22).

The factor 4/3 is the same as in Lorentz's expression and is in contradiction with the considerations based on the energy conservation. One way to understand its origin is the note that the rigidly prescribed charge distribution, used in these early calculations before 1905 violates special relativity in the absence of Lorentz contraction.

Approximately in the same time Sommerfeld calculated the self force acting on a charge distribution $\rho(\mathbf{x})$ in its co-moving coordinate system by ignoring the higher order than linear terms in the acceleration and its time derivatives [9],

$$\mathbf{F}_{rad} = \frac{2}{3}e^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}c_n\frac{d^n}{dt^{n+1}}\mathbf{v}, \tag{7.30}$$

where

$$c_n = \int d^3xd^3y\rho(\mathbf{x})|\mathbf{x}-\mathbf{y}|^n\rho(\mathbf{y}). \tag{7.31}$$

He considered charges distributed homogeneously on the surface of a sphere of radius r_0 when

$$c_n = \frac{1}{(2r_0)^{n-1}}\frac{2}{n+1} \tag{7.32}$$

and managed to resum the series which results the non-relativistic equation of motion

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_{ext} + m_{ed}\frac{\mathbf{v}(t-2a)-\mathbf{v}(t)}{2r_0} \tag{7.33}$$

which is a finite difference equation, the delay time needed to reach the opposite points of the sphere.

Soon after the discovery of special relativity Laue has found the relativistic extension of Lorentz's result (7.27), [10]

$$m\ddot{x}^\mu = -\frac{2}{3}e^2(\dot{x}^\mu\ddot{x}^2 + \dddot{x}^\mu), \tag{7.34}$$

an equation to be derived in a more reliable manner later by Dirac. The first term is positive definite and represents the breakdown of the time inversion invariance, the loss of energy due to the 'friction' caused by the radiation. The second, the so called Schott term can change sign and stands for the emission and absorption processes.

A particularly simple, phenomenological argument to arrive at the self force (7.34) [11] is based on the constraint $\ddot{x}\cdot\dot{x}=0$ on the world line of a point particle which asserts that the four-force $F^\mu_{rad} = m_{mech}\ddot{x}^\mu$ must be orthogonal to the four-velocity. One can easily construct the linearized equation of motion, an orthogonal vector which is linear in the velocity and its derivatives by means of the projector

$$P^{\mu\nu} = g^{\mu\nu} - \dot{x}^\mu\dot{x}^\nu \tag{7.35}$$

as

$$F^\mu_{rad} = P^{\mu\nu}\sum_{n=1}^\infty a_n \frac{d^n x^\nu(s)}{ds^n}. \tag{7.36}$$

The derivative of the constraint $\ddot{x}\cdot\dot{x} = 0$ with respect to the world-line parameter,

$$\dddot{x}\cdot\dot{x} + \ddot{x}^2 = 0, \tag{7.37}$$

gives for the self force truncated at the third derivative

$$m_{mech}\ddot{x}^\mu \approx a_2\ddot{x}^\mu + a_3(\dddot{x}^\mu + \dot{x}^\mu\ddot{x}^2). \tag{7.38}$$

The mass renormalization $m = m_{mech} - a_2$ eliminates the first term on the right hand side and expresses the physical mass as the sum of the mechanical and part and the contribution from electrodynamics. The comparison with Eq. (7.34) gives $a_3 = -2e^2/3$.

The third derivative with respect to the time in the equation of motion presents a new problem, such an equation has runaway, self-accelerating solution,

$$\dot{x}^0 = \cosh[r_0(e^{s/r_0} - 1)], \quad \dot{x}^1 = \sinh[r_0(e^{s/r_0} - 1)], \quad \dot{x}^2 = \dot{x}^3 = 0, \tag{7.39}$$

with r_c being the classical electron radius (7.24). This is unacceptable. Dirac proposed an additional boundary condition in time for the charges which is needed for the equation of motion with third time derivative. This is to be imposed at the final time and it eliminates the runaway solutions. The problem which renders this proposal difficult to accept is that it generates acausal effects on the motion of the charges, acceleration before the application of the forces.

The origin of this problem is the sharp boundary of the homogeneously distributed charge in a sphere when the radius r_0 tends to zero. It was shown that both the runaway and the preaccelerating solutions are absent for Sommerfeld's equation of motion (7.33) as long as $r_0 > r_{cl}$ [12]. It is the truncated power series approximation for this finite difference equation which creates the problem in the point charge limit.

7.4.2 Point charge limit

We present here a part of Dirac's work [13] which aims at the determination of the equation of motion of a charged body with localized but otherwise unspecified distribution. Dirac decided not to use the Lorentz force and the energy-momentum conservation is his only dynamical input which can be applied safely, outside of the body. The calculation is based on the introduction of a tube of radius r_0 around the world-line of the body which is supposed to include all charge and the calculation of the energy-momentum balance equation for the charge-electromagnetic field interactive system. It turns out that the forces acting on the body can be determined and are local in time when the limit $r_0 \to 0$ is taken. The equation of motion obtained in this manner is not relying on the Lorentz force, a singular construction for point charges.

We follow here a shorter path to reach the goal [4], the direct use of the Lorentz force and a careful execution of the point limit. Such a calculation alone is not completely convincing but once the same result is obtained as by the safer but more involved balance equation one can accept the argument which at the end provides more insight.

First let us introduce the manifestly Lorentz-covariant separation of the near and far fields, $A = D^r \cdot j + A_{in} = D^a \cdot j + A_{out}$

$$\begin{aligned} A_{rad} &= A_{out} - A_{in} = A^r - A^a \\ A^r &= \underbrace{\frac{1}{2} A_{rad}}_{A^f} + \underbrace{\frac{1}{2}(A^r + A^a)}_{A^n}. \end{aligned} \tag{7.40}$$

He found that the correction force to the equation of motion comes entirely from A^f which is finite and regular at the point charge and the near field is responsible of the divergences arising in the point charge limit. The actual calculation is subtle because the emission of radiation does not commute with the limit $r_0 \to 0$ since the radiation is constrained onto the future light-cone which can not be pierced by the world-line of a massive particle. In other words, a strictly point charge can not have back reaction force, this latter comes entirely from $r_0 > 0$.

We start with

$$F^{\mu\nu}_{rad}(s') = 4\pi e \int ds D_{rad}(x - x') \frac{d}{ds} \frac{(x - x(s))^\mu \dot{x}^\nu(s)}{(x - x(s)) \cdot \dot{x}(s)} - (\mu \longleftrightarrow \nu), \tag{7.41}$$

where

$$D_{rad}(x) = D^r(x - x') - D^a(x - x') = \frac{1}{2\pi} \epsilon(x^0 - x'^0) \delta((x - x')^2). \tag{7.42}$$

We write $s = s' + u$ and expand in u,

$$
\begin{aligned}
x(s) - x(s') &= u\dot{x} + \frac{u^2}{2}\ddot{x} + \frac{u^3}{6}\frac{d^3x}{ds^3} + \cdots \\
\dot{x}(s) &= \dot{x} + u\ddot{x} + \frac{u^2}{2}\frac{d^3x}{ds^3} + \cdots \\
(x(s) - x(s'))^2 &= u^2 + \mathcal{O}\left(u^4\right) \\
(x(s) - x(s')) \cdot \dot{x} &= u + \mathcal{O}\left(u^3\right) \\
\epsilon(x^0(s) - x'^0(s')) &= \epsilon(u) \qquad (7.43)
\end{aligned}
$$

to find

$$
\begin{aligned}
F^{\mu\nu}_{rad}(s') &= 2e \int ds \epsilon(u) \\
&\quad \times \delta(u^2)\frac{d}{du}\left[\left(\dot{x} + \frac{u}{2}\ddot{x} + \frac{u^2}{6}\frac{d^3x}{ds^3}\right)^\mu \left(\dot{x} + u\ddot{x} + \frac{u^2}{2}\frac{d^3x}{ds^3}\right)^\nu\right] \\
&\quad -(\mu \longleftrightarrow \nu) \qquad (7.44) \\
&= 2e \int du \epsilon(u)\delta(u^2) \\
&\quad \frac{d}{du}\left(u\dot{x}^\mu\ddot{x}^\nu + \frac{u}{2}\ddot{x}^\mu\dot{x}^\nu + \frac{u^2}{2}\dot{x}^\mu\frac{dx^\nu}{ds^3} + \frac{u^2}{6}\frac{dx^\mu}{ds^3}\dot{x}^\nu\right) - (\mu \longleftrightarrow \nu).
\end{aligned}
$$

The small but finite size of the charge compared with the width of the light cone where the radiation field is constrained is taken into account by the formal steps

$$
\delta(u^2) = \lim_{v\to 0^+} \delta(u^2 - v^2) = \lim_{v\to 0^+}\left[\frac{\delta(u-v)}{2v} + \frac{\delta(u+v)}{2v}\right] \qquad (7.45)
$$

and

$$
\epsilon(u)\delta(u^2) = \lim_{v\to 0^+}\left[\frac{\delta(u-v)}{2v} - \frac{\delta(u+v)}{2v}\right] = -\delta'(u), \qquad (7.46)
$$

yielding

$$
\begin{aligned}
F^{\mu\nu}_{rad}(s') &= 2e \int du \delta(u)\frac{d^2}{du^2}\left(\frac{u}{2}\dot{x}^\mu\ddot{x}^\nu + \frac{u^2}{3}\dot{x}^\mu\frac{dx^\nu}{ds^3}\right) - (\mu \longleftrightarrow \nu) \\
&= \frac{4}{3}e\left(\dot{x}^\mu\frac{d^3x^\nu}{ds^3} - \dot{x}^\nu\frac{d^3x^\mu}{ds^3}\right) \qquad (7.47)
\end{aligned}
$$

and

$$
\begin{aligned}
K^\mu_{react} = mc\ddot{x}^\mu_{rad}(x') &= \frac{e}{2c}F^{\mu\nu}_{rad}(x')\dot{x}_\nu \\
&= \frac{2}{3}e^2(\dot{x}^\mu\dddot{x}\cdot\dot{x} - \dddot{x}^\mu). \qquad (7.48)
\end{aligned}
$$

Finally, by the the use of Eq. (7.37) we recover Eq. (7.34). The regularization of the product of two distributions in Eq. (7.46) is the delicate point whose only satisfactory verification is the recovery of the already known reaction force.

The near field represents no loss or gain in energy and momentum, it rather enriches the structure of the charge by modifying, renormalizing its free equation of motion. Dirac found that Lorentz's divergent m_{react} is given by the near-field and gives rise a mass renormalization. To see this we start with the action

$$\begin{aligned} S &= -m_b c \int ds + S_{ed} \\ S_{ed} &= -\frac{e}{c} \int d^4x A^\nu(x) j_\nu(x) \\ &= -\frac{e}{2} \int d^4x [A^r_\nu(x) + A^a_\nu(x)] \int ds \underbrace{\rho(x - x(s))}_{\text{form factor}} \dot{x}^\nu(s) \end{aligned} \tag{7.49}$$

We write the near field $A^n_\nu(x) = \frac{1}{2}[A^r_\nu(x) + A^a_\nu(x)]$ as

$$\begin{aligned} A^n_\nu(x) &= 4\pi e \int d^4x' ds' \frac{1}{2} [D^r(x - x') + D^{adv}(x - x')] \rho(x' - x(s')) \dot{x}_\nu(s') \\ &= e \int d^4x' ds' \delta((x - x')^2) \rho(x' - x(s')) \dot{x}_\nu(s') \end{aligned} \tag{7.50}$$

which yields, upon inserted into the action

$$\begin{aligned} S_{ed} &= -e^2 \int d^4x d^4x' ds ds' \delta((x - x')^2) \rho(x' - x(s')) \rho(x - x(s)) \dot{x}(s') \cdot \dot{x}(s) \\ &= -e^2 \int d^4w d^4w' ds ds' \\ &\quad \times \delta((w - w' + x(s) - x(s'))^2) \rho(w') \rho(w) \dot{x}(s') \cdot \dot{x}(s). \end{aligned} \tag{7.51}$$

This was the decisive step, this action does not contain the Liénard-Wiechert potentials anymore, the Green functions were used to eliminate the electromagnetic field from the problem by means of their equations of motion. We follow the limit $r_0 \to$, $s' = s + u$, $\dot{x}(s') = \dot{x} + u\ddot{x} + \cdots$, $x(s) - x(s') = -u\dot{x} + \cdots$

and write

$$
\begin{aligned}
S_{ed} &\approx -e^2\int dsd^4wd^4w'du\delta((w-w'-u\dot{x})^2)\rho(w')\rho(w)\\
&= -\frac{e^2}{2}\int dsd^4wd^4w'\\
&\quad\times\left[\frac{1}{(w-w'-u_{ret}\dot{x})\cdot\dot{x}}+\frac{1}{(w-w'-u_{adv}\dot{x})\cdot\dot{x}}\right]\rho(w')\rho(w)\\
&= -\frac{e^2}{2}\int dsd^4wd^4w'\\
&\quad\times\left[\frac{1}{(w-w')\cdot\dot{x}-u_{ret}}+\frac{1}{(w-w')\cdot\dot{x}-u_{adv}}\right]\rho(w')\rho(w)\\
&= -m_{ed}c\int ds \qquad (7.52)
\end{aligned}
$$

where the integral in the third equation diverges and becomes independent of s when $\rho(x)\to\delta^4(x)$ and

$$
m_{ed}=\frac{e^2}{2c}\int d^4wd^4w'\left[\frac{1}{(w-w')\cdot\dot{x}-u_{ret}}+\frac{1}{(w-w')\cdot\dot{x}-u_{adv}}\right]\rho(w')\rho(w) \qquad (7.53)
$$

What is found is a renormalization of the mass, the combination $m_{ph}=m_b+m_{el}$ is observable only which sets $m_b=m_{ph}-m_{ed}$.

7.4.3 Iterative solution

The coupled equations of motion for the charge and the electromagnetic field can be solved iteratively, by reinserting the Liénard-Wiechert potential obtained by means of the solution of the mechanical equation [2]. They set up a perturbation expansion in the retardation which comprises the nontrivial effects of the radiation and obtain the radiation force in two steps. First they calculate the effective Lagrangian for the charge, obtained by eliminating the electromagnetic field by the Maxwell equation in order $\mathcal{O}(\mathbf{v}/c)$. The next order contain the radiational friction force and is obtained by iterating the equation of motion. It is reassuring to see that the further iterations in the retardation yield vanishing result in the point charge limit.

The retarded Liénard-Wiechert potential (which can not be obtained from an action principle due to its non time reflection symmetrical form) leads to the effective Lagrangian

$$
\begin{aligned}
L &= -\sum_a m_a c^2 \sqrt{1-\frac{v_a^2}{c^2}} - \sum_a e_a \phi + \sum_a \frac{e_a}{c}\mathbf{A}\cdot\mathbf{v}_a \\
&= \sum_a \left[\frac{m_a v_a^2}{2} + \frac{m_a v_a^4}{8c^2} + \mathcal{O}\left(\frac{v^6}{c^6}\right)\right] - \sum_a e_a \int d^3x' \frac{\rho(t-\frac{|\mathbf{x}_a-\mathbf{x}'|}{c},\mathbf{x}')}{|\mathbf{x}_a-\mathbf{x}'|} \\
&\quad + \sum_a \frac{e_a}{c^2}\int d^3x' \frac{\mathbf{j}(t-\frac{|\mathbf{x}_a-\mathbf{x}'|}{c},\mathbf{x}')}{|\mathbf{x}_a-\mathbf{x}'|}\cdot\mathbf{v}_a \qquad (7.54)
\end{aligned}
$$

for a system of charges when the self-interaction is retained.

We make an expansion in the retardation by assuming $v \ll c$, $R/c \ll \tau$, τ being the characteristic time scale of the charges. Note that the factor $|\mathbf{x_a}-\mathbf{x}'|^n$ in the higher, $\mathcal{O}\left((\frac{v^2}{c^2})^n\right)$ order contributions with $n \geq 3$ suppresses the singularity at $|\mathbf{x_a}-\mathbf{x}'| = 0$. We find

$$
\begin{aligned}
\phi(t,\mathbf{r}_a) &= \sum_b \Bigg[\int dr_b \frac{\rho(t,\mathbf{r}_b)}{R_{ab}} - \frac{1}{c}\partial_t \underbrace{\int dr_b \rho(t,\mathbf{r}_b)}_{Q=\text{const.}} + \frac{1}{2c^2}\partial_t^2 \int dr_b R_{ab}\rho(t,\mathbf{r}_b) \\
&\quad - \frac{1}{6c^3}\partial_t^3 \int dr_b R_{ab}^2 \rho(t,\mathbf{r}_b)\Bigg] + \mathcal{O}\left(\frac{1}{c^4}\right) \\
\mathbf{A}(t,\mathbf{r}_a) &= \sum_b \left[\frac{1}{c}\int dr_b \frac{\mathbf{j}(t,\mathbf{r}_b)}{R_{ab}} - \frac{1}{c^2}\partial_t \int dr_b \mathbf{j}(t,\mathbf{r}_b)\right] + \mathcal{O}\left(\frac{1}{c^3}\right), \qquad (7.55)
\end{aligned}
$$

what yields

$$
\begin{aligned}
\phi_a &= \sum_b \left[\frac{e_b}{R_{ab}} + \frac{e_b}{2c^2}\partial_t^2 R_{ab} - \frac{e_b}{6c^3}\partial_t^3 R_{ab}^2\right] \\
\mathbf{A}_a &= \sum_b \left[\frac{e_b \mathbf{v}_b}{cR_{ab}} - \frac{e_b}{c^2}\partial_t \mathbf{v}_b\right] \qquad (7.56)
\end{aligned}
$$

in the point charge limit. We perform the gauge transformation

$$
\begin{aligned}
\phi'_a &= \phi_a - \frac{1}{c}\sum_b \left[\partial_t\left(\frac{e_b}{2c}\partial_t R_{ab} - \frac{e_b}{6c^2}\partial_t^2 R_{ab}^2\right)\right] = \sum_b \frac{e_b}{R_{ab}} = \phi_a'^{(0)} \\
\mathbf{A}'_a &= \mathbf{A}_a + \nabla \sum_b \left[\left(\frac{e_b}{2c}\partial_t R_{ab} - \frac{e_b}{6c^2}\partial_t^2 R_{ab}^2\right)\right] \\
&= \sum_b \left[\frac{e_b \mathbf{v}_b}{cR_{ab}} - \frac{e_b}{c^2}\partial_t \mathbf{v}_b + \frac{e_b}{2c}\nabla\partial_t R_{ab} - \frac{e_b}{6c^2}\partial_t^2 \underbrace{\nabla R_{ab}^2}_{2\mathbf{R}_{ab}}\right] = \mathbf{A}_a'^{(1)} + \mathbf{A}_a'^{(2)} \qquad (7.57)
\end{aligned}
$$

The Lagrangian is

$$L^{(0)} = \sum_a \frac{m_a \mathbf{v}_a^2}{2} - \frac{1}{2} \sum_{a \neq b} \frac{e_a e_b}{R_{ab}} \tag{7.58}$$

in the non-relativistic limit, $\mathcal{O}\left((\frac{v}{c})^0\right)$ after ignoring an unimportant, diverging self energy for $a = b$.

For the next non-relativistic order, $\mathcal{O}\left(\frac{v}{c}\right)$, we need

$$\nabla \partial_t R = \partial_t \nabla R = \partial_t \mathbf{n} = \frac{\partial_t \mathbf{R}}{R} - \frac{\mathbf{R} \partial_t R}{R^2} \tag{7.59}$$

where $\mathbf{n} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ denotes the unit vector from the charge to the observation point and $R \partial_t R = \mathbf{R} \partial_t \mathbf{R} = -\mathbf{R} \mathbf{v}$, with

$$\partial_t \mathbf{n} = \frac{-\mathbf{v} + \mathbf{n}(\mathbf{n} \cdot \mathbf{v})}{R}. \tag{7.60}$$

One finds

$$\begin{aligned} \phi_a'^{(0)} &= \sum_b \frac{e_b}{R_{ab}} \\ \mathbf{A}_a'^{(1)} &= \sum_b e_b \frac{\mathbf{v}_b + \mathbf{n}_b(\mathbf{n}_b \cdot \mathbf{v}_b)}{2cR_{ab}} \end{aligned} \tag{7.61}$$

and the Lagrangian is

$$\begin{aligned} L^{(2)} &= \sum_a \left(\frac{m_a v_a^2}{2} + \frac{m_a v_a^4}{8c^2} \right) \\ &\quad - \frac{1}{2} \sum_{a \neq b} \frac{e_a e_b}{R_{ab}} + \frac{1}{2} \sum_{a \neq b} \frac{e_a e_b}{c^2 R_{ab}} [\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{v}_a \cdot \mathbf{n}_{ab})(\mathbf{v}_b \cdot \mathbf{n}_{ab})] \end{aligned} \tag{7.62}$$

in this order after a diverging self energy is ignored again for $a = b$.

The next, $\mathcal{O}\left((\frac{v}{c})^2\right)$ order electromagnetic field contains the radiation induced friction force and can not be represented in the Lagrangian. We set $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\partial_t \mathbf{R} = -\partial_t \mathbf{r}'$ and write

$$\begin{aligned} \mathbf{A}_a'^{(2)} &= \sum_b \left[\frac{e_b}{c^2} \partial_t \mathbf{v}_b + \frac{e_b}{3c^2} \partial_t \mathbf{v}_b \right] \\ &= -\sum_b \frac{2}{3} \frac{e_b}{c^2} \partial_t \mathbf{v}_b. \end{aligned} \tag{7.63}$$

In the absence of explicit $\mathbf{x}$-dependence the magnetic field is vanishing in this order, $\mathbf{H}^{(2)} = 0$. The force acting on the charge is of electric origin alone and the self force arises from the electric field

$$\mathbf{E}_a^{(2)} = -\frac{1}{c}\partial_t \mathbf{A}_a'^{(2)} - \nabla \underbrace{\phi_a^{(2)}}_{=0} = \frac{2}{3}\frac{e_a \partial_t^3 \mathbf{x}_a}{c^3} \tag{7.64}$$

The energy loss per unit time is

$$\begin{aligned} W &= \sum_a \mathbf{F}_a \cdot \mathbf{v}_a \\ &= \frac{3}{2}\frac{1}{c^3}\sum_b e_b \partial_t^3 \mathbf{x}_b \cdot \sum_a e_a \partial_t \mathbf{x}_a \\ &= \frac{3}{2}\frac{1}{c^3}\sum_{ab} e_a e_b [\partial_t(\partial_t^2 \mathbf{x}_b \cdot \partial_t \mathbf{x}_a) - (\partial_t^2 \mathbf{x}_a \cdot \partial_t^2 \mathbf{x}_b)] \end{aligned} \tag{7.65}$$

with the time average

$$\overline{W} = -\frac{3}{2}\frac{1}{c^3}\sum_{ab} e_a e_b \overline{(\partial_t^2 \mathbf{x}_a \cdot \partial_t^2 \mathbf{x}_b)} \tag{7.66}$$

where the total derivative term can be neglected.

The higher order contributions in the retardation become negligible in the point-like charge limit when $\mathbf{R} \to 0$ and the expression for the radiation reaction force

$$\mathbf{F}_{rr} = \frac{2}{3}\frac{e^2 \partial_t^2 \mathbf{v}}{c^3} \tag{7.67}$$

becomes exact! We see that we recover the second term in the right hand side of the last equation of Eqs. (7.34) but not the first one in this manner, by relying on the retarded potentials.

The non-relativistic equation of motion,

$$m\partial_t \mathbf{v} = \frac{2}{3}\frac{e^2 \partial_t^2 \mathbf{v}}{c^3} \tag{7.68}$$

leads unavoidable to the runaway solution

$$\partial_t \mathbf{v} = \mathbf{v}_0 e^{t\frac{3}{2}\frac{mc^3}{e^2}}. \tag{7.69}$$

The equation of motion with the Lorentz-force, corrected by the radiation reaction is

$$m\partial_t \mathbf{v} = e\mathbf{E}_{ext} + \frac{e}{c}\mathbf{v} \times \mathbf{H}_{ext} + \frac{2}{3}\frac{e^2 \partial_t^2 \mathbf{v}}{c^3}. \tag{7.70}$$

We arrived finally at a central question: at what length scales can we see the radiation reaction forces? The condition for the radiation back-reaction be small and an iterative solution is applicable is the following. In the rest frame

$$\partial_t^2 \mathbf{v} = \frac{e}{m}\partial_t \mathbf{E}_{ext} + \frac{e}{mc}\partial_t \mathbf{v} \times \mathbf{H}_{ext} + \mathcal{O}\left(c^{-3}\right). \tag{7.71}$$

Since $\partial_t \mathbf{v} = e\mathbf{E}_{ext}/m$,

$$\partial_t^2 \mathbf{v} = \frac{e}{m}\partial_t \mathbf{E}_{ext} + \frac{e^2}{m^2 c}\mathbf{E}_{ext} \times \mathbf{H}_{ext} + \mathcal{O}\left(c^{-3}\right) \tag{7.72}$$

and the radiation reaction force is

$$\mathbf{F}_{rr} = \underbrace{\frac{2}{3}\frac{e^3}{mc^3}\partial_t \mathbf{E}_{ext}}_{\mathcal{O}\left(\frac{e^3\omega}{mc^3}E\right)} + \underbrace{\frac{2}{3}\frac{e^4}{m^2c^4}\mathbf{E}_{ext} \times \mathbf{H}_{ext}}_{\mathcal{O}\left(\frac{e^4}{m^2c^4}EH\right)} + \mathcal{O}\left(c^{-5}\right). \tag{7.73}$$

The first term is negligible compared with the force generated by the external electric field for a monochromatic field with frequency ω if

$$\frac{|\mathbf{F}_{rr}|}{|\mathbf{F}_{ext}|} \approx \frac{e^2\omega}{mc^3} \ll 1 \tag{7.74}$$

or

$$\frac{e^2}{mc^2} \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}. \tag{7.75}$$

Thus classical electrodynamics becomes inconsistent due to pair creations at distances shorter than the classical charge radius, $\ell \approx \lambda_C = e^2/mc^2$.

We note that the second term is negligible,

$$H \ll \frac{m^2c^4}{e^3} \tag{7.76}$$

for realistic magnetic fields.

7.4.4 Action-at-a-distance

A different approach to electrodynamics which might be called effective theory in the contemporary jargon is based on the elimination of the electromagnetic field altogether from the theory [14, 15, 16].

Let us write the action of a system of charges described by their world lines $x_a^\mu(s)$ and the electromagnetic field in a condensed notation as

$$S = \sum_a S_m[x_a] + \frac{1}{2}A \cdot D^{-1} \cdot A - \sum_a j_a \cdot A \tag{7.77}$$

where the dot stands for space-time integration and index summation, $j \cdot A = \int dx j^\mu(x) A_\mu(x)$, etc. The Maxwell equation, $\frac{\delta S}{\delta A} = 0$, yields

$$A = D \cdot j. \tag{7.78}$$

This equation can be used to eliminate A from the action and to construct the effective theory for the charges with the action

$$\begin{aligned} S &= \sum_a S_m[x_a] + \frac{1}{2} \sum_{ab} j_a \cdot D \cdot D^{-1} \cdot D \cdot j_b - \sum_{ab} j_a \cdot D \cdot j_b \\ &= \sum_a S_m[x_a] - \frac{1}{2} \sum_{ab} j_a \cdot D^{-1} \cdot j_b \\ &\to \sum_a S_m[x_a] - \frac{1}{2} \sum_{a \neq b} j_a \cdot D^{-1} \cdot j_b \end{aligned} \tag{7.79}$$

without the electromagnetic field. The elimination of the field degrees of freedom generates action-at-a-distance. The self-interaction was omitted in the last equation.

The Maxwell equation indicates that D should be a Green-function. But which one? According to Dirac's proposal we have near and far field Green-functions

$$A^{\overset{n}{f}} = \frac{1}{2}(A^r \pm A^a) = \frac{1}{2}(D^r \pm D^a) \cdot j \tag{7.80}$$

which motivates the notation $D^{\overset{n}{f}} = \frac{1}{2}(D^r \pm D^a)$. Whatever Green-function we use, the symmetric part survives only because $A \cdot B \cdot A = 0$ for an antisymmetrical operator, $B^{\text{tr}} = -B$. Since $D^a(x, y) = D^r(y, x)$, D^n and D^f are just the symmetric and antisymmetric part of the inhomogeneous propagator and we have to use D^n in the action principle. The self-interaction generated by the near-field and ignored in the last line of Eqs. (7.79) is indeed a world-line independent, divergent term.

The support of the Green-function is the light-cone therefore the charge a at point x_a interacts with the charge b if the world-line $x_b(s)$ of the charge b pierces the light-cone erected at point x_a. The interaction is governed by the near-field Green-function and it is 50% retarded and 50% advanced. Such an even distribution of the retarded and advanced interaction assures the formal time inversion invariance.

The unwanted complication of the near-field mediated interactions is that it eliminates radiation field and the retardation effects. It is a quite cumbersome procedure to add by hand the appropriate free field to the solution which restores the desired initial conditions.

The use of the retarded Green-function assumes that the in-fields are weak. This is not the case for the out-fields and the time inversion symmetry is broken. A sufficient plausible assumption to explain this phenomenon is the proposition that the Universe is completely absorptive, there is no electromagnetic radiation reaching spatial infinities due to the elementary scattering processes of the inter-galactic dust [17]. The equation of motion for the charge a in the theory given by the action (7.79),

$$\begin{aligned} mc\ddot{x}^{\mu}_{a} &= \frac{e}{c}F^{n\mu\nu}\dot{x}_{a\nu} \\ &= \frac{e}{2c}\sum_{b\neq a}(F^{r\mu\nu}_{b} + F^{a\mu\nu}_{b})\dot{x}_{a\nu} \end{aligned} \tag{7.81}$$

can be written as

$$mc\ddot{x}^{\mu}_{a} = \frac{e}{c}\left[\sum_{b\neq a}F^{r\mu\nu}_{b} + \frac{1}{2}(F^{r\mu\nu}_{a} - F^{a\mu\nu}_{a}) - \frac{1}{2}\sum_{b}(F^{r\mu\nu}_{b} - F^{a\mu\nu}_{b})\right]\dot{x}_{a\nu}. \tag{7.82}$$

The first term represents the usual retarded interaction with the charges, self interactions ignored. The second term is the regular far field generated by the charge and provides the forces needed for the energy-momentum conservation for radiating charges. The last expression, the radiation field of all charges is vanishing in a completely absorbing Universe. The origin of the breakdown of the time reversal invariance needed for the appearance of the radiation friction force which can be derived without difficulty from Eq. (7.82) is thus located in the absorbing nature of the Universe. Calculations performed in Quantum Electrodynamics in finite, flat space-time support the absorbing Universe hypothesis.

7.4.5 Beyond electrodynamics

Similar radiation back-reaction problem exist in any interactive particle-field theory, for instance gravity or a more academic model where the interaction is mediated by a massive scalar field.

A mass curves the space-time around itself and is actually moves in such a distorted geometry. part of the distortion is instantaneous, the analogy of the Coulomb force of electrodynamics, another part displays retardation and represents gravitational radiation. It was found [18, 19] that there is indeed a radiation back-reaction force in gravity and its additional feature is that it has a non-local component parallel to the four-acceleration, hence the mass is renormalized by a term which depends on the whole past of the motion.

It is the special vector algebra which rendered the mass renormalization a part and time independent constant in Eq. (7.22) for the electromagnetic interaction in flat space-time. But a conceptual issue which remains to settle in the gravitational case is that in general any explicit use of the space-time coordinate corresponds to a gauge choice, in particular the form of the self force one can get is gauge dependent and not physical. The satisfactory solution of this problem which is still ahead of us is to translate all relevant dynamical issues into gauge invariant, coordinate choice independent form. The loss of the mass as a constant to characterize the motion of a point particle obviously forces radical changes upon our way to imagine classical physics.

The origin of the non-local nature of the self-force can easily be understood. An external background curvature acts as a mass term for the gravitational radiation. Therefore, the dynamical problem here is like the radiation back-reaction arising from interacting with a massive field. This problem can specially easily analyzed in the case of a massive scalar field. Its retarded Green function is non-vanishing within the future light cone as opposed to the massless Green function whose support is the future light cone only. Therefore whole past of the world-line for the point $x(s)$ lies within the past light cone of $x(s)$ and contribute to the self-force as opposed to the simple situation of the massless electromagnetic interaction, depicted in Fig. 6.1.

7.5 Epilogue

The recent developments in High Energy Physics, namely the construction of effective theories based on the use of the renormalization group shows clearly the origin of the Abraham-Lorentz force. When degrees of freedom are eliminated in a dynamical system by means of their equation of motion then the equations of motion of the remaining degrees of freedom change. The new terms represent the correlations realized by the eliminated degrees of freedom in the dynamics of the remaining part of the system. When the effect of the self field is considered on a charge then we actually eliminate the EM field and generate new pieces to the equations of motion for the charges. These are the radiation back-reaction forces, their importance can systematically be estimated by the method of the renormalization group, applied either on the classic or the quantum level.

It may happen that the coupled set of equations of motion for point charges and the electromagnetic field has no regular solution and electrodynamics of point charges is well defined on the quantum level only. This is reminescent of the possibility that the proof of relaxation and the approach to an equilibrium ensemble in statistical physics might need quantum mechanics.

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