Research Article

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A note on the differences between Drucker-Prager and Mohr-Coulomb shear strength criteria

Abstract: A systematic approach to measure the differences between Mohr-Coulomb (MC) and Drucker-Prager (DP) shear strength criteria used commonly in soil and rock mechanics is presented. It is shown that the DP criterion generates a shear strength between 0.6 and 3 times the MC strength, for the same friction angle and cohesion parameters. The appropriate conditions for obtaining equal shear strengths are given. Moreover, some new DP failure surfaces are proposed which minimize the differences relative to the MC predictions. The equivalence of the DP and MC criteria under plane strain conditions is also examined.

Keywords: Mohr-Coulomb; Drucker-Prager; elasto-plasticity; shear strength; plane strain conditions.

1 Introduction

In soil and rock mechanics, the Mohr-Coulomb (MC) shear strength criterion, along with its parameters, namely friction angle and cohesion, is treated as a kind of standard and reference concept for other shear strength criteria. This is due to the fact, that it fits well the experimental data, where asymmetric strength response in triaxial compression (TXC) and triaxial extension (TXE) tests is observed. Moreover, the MC criterion parameters have clear engineering interpretation and they are typically obtained in most geotechnical laboratories. On the other hand the MC concept discards the influence of the intermediate principal stress on the shear strength of the material, whereas this influence is visible when true triaxial testing is performed [1, 9]. Additionally, the MC failure surface, defined in the principal stress space, contains sharp edges, which introduces some difficulties

in the implementation of the criterion for numerical analysis purposes. These shortcomings are often opposed to the straightforward implementation of the smooth failure functions, with Drucker-Prager (DP) criterion being one of such examples [2, 7].

Of continuous interest to researchers is the question regarding how the different shear strength critera used in engineering and computational practice compare to the reference MC predictions. For this purpose, a concept of equivalent friction angle is usually used [3, 5, 6, 8]. This angle is defined as the friction angle of the MC surface that would pass through the particular stress point given by the shear strength criterion under consideration. Its variations with the changing stress state and the parameters of the criterion being compared are then analysed.

In this paper, another approach is used. Instead of defining and analysing the equivalent friction angle, the shear strengths predicted by the MC and DP criteria are compared directly. In case of these particular criteria, an analytical formula can be derived for this purpose. To the knowledge of the author, such a formula has not been published explicitly yet. From this result, it is deduced that the DP criterion generates shear strength between 0.6 and 3 times the MC strength, for the same friction angle and cohesion parameters. The appropriate conditions for obtaining equal strengths for both criteria are also analysed. Additionally, some new DP failure surfaces that minimize differences with MC criterion are proposed.

2 Basic notation

Let us assume that the principal stresses in the isotropic material are given by $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and let us assume the positive sign for compressive stresses. The invariants of the stress state used in the following text can be written as expressed in Eqs (1)-(3):

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$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \tag{1}$$

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2], \tag{2}$$

$$J_3 = \frac{1}{81}[(2\sigma_1 - \sigma_2 - \sigma_3)^3 + (2\sigma_2 - \sigma_1 - \sigma_3)^3 + (2\sigma_3 - \sigma_1 - \sigma_2)^3], \tag{3}$$

where I_1 is the first invariant of the stress tensor and J_2, J_3 are the second and the third invariants of the stress deviator. The intermediate principal stress can always be represented as a linear combination of two other stresses:

$$\sigma_2 = (1 - b)\sigma_1 + b\sigma_3, \tag{4}$$

where:

$$b = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} \in \langle 0,1 \rangle \tag{5}$$

is the so called principal stress ratio. Let us now introduce the maximum shear plane stresses[1], i.e.:

$$p = \frac{\sigma_1 + \sigma_3}{2}, \tag{6}$$

$$q = \frac{\sigma_1 - \sigma_3}{2}. \tag{7}$$

The principal stresses can be now written as follows:

$$\sigma_1 = p + q, \tag{8}$$
$$\sigma_2 = p - aq, \tag{9}$$
$$\sigma_3 = p - q, \tag{10}$$

where:

$$a = 2b - 1 \in \langle -1,1 \rangle \tag{11}$$

is an equivalent measure of principal stress ratio. Introducing equations (8)-(10) into the definitions of invariants the following relations are obtained:

$$I_1 = 3p - aq, \tag{12}$$

$$J_2 = \frac{q^2}{3}(a^2 + 3), \tag{13}$$

$$J_3 = \frac{2a}{27}q^3(-a^2 + 9). \tag{14}$$

Finally the Lode angle $\theta \in \langle -\pi/6, \pi/6 \rangle$ is introduced by Eq. (15):

$$\sin(3\theta) = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{\frac{3}{2}}. \tag{15}$$

It is straightforward to show that Lode angle is related to the parameter a via Eq. (16):

$$a = \sqrt{3}\tan\theta \tag{16}$$

and it can be viewed as yet another measure of the principal stress ratio. For all three measures namely b, a, θ, the left limit of their values stands for the $\sigma_1 = \sigma_2$ case, i.e. triaxial extension (TXE), whereas the right limit signifies $\sigma_2 = \sigma_3$, i.e. triaxial compression (TXC).

In the following sections p, q and θ will be used as the representation of the stress states in the material.

3 MC and DP shear strength criteria

The MC strength criterion can be written in the following form:

$$q^{MC} = p\sin\phi + c\cos\phi, \tag{17}$$

where ϕ is the friction angle and c is the material cohesion. It is quite obvious that the failure shear stress q^{MC} does not depend on θ, which means also that the intermediate stress σ_2 does not influence material shear strength. On the other hand the DP condition is usually expressed via the following relation:

$$\sqrt{J_2} + \alpha I_1 + k = 0, \tag{18}$$

where α and k are parameters of this criterion. Inserting Eqs. (12), (13) and (16) into Eq. (18) the expression for the failure shear stress in the DP condition is derived as following:

$$q^{DP} = \frac{(3\alpha p + k)\cos(\theta)}{\sqrt{3}\alpha\sin(\theta) - 1}. \tag{19}$$

Comparison of the free coefficients and the coefficients standing by p in these two criteria leads us to the expressions for α and k in relation to ϕ and c:

$$\alpha = \frac{\sin(\phi)}{\sqrt{3}\sin(\phi)\sin(\theta) - 3\cos(\theta)}, \tag{20}$$

$$k = \frac{3c\cos(\phi)}{\sqrt{3}\sin(\phi)\sin(\theta) - 3\cos(\theta)}. \tag{21}$$
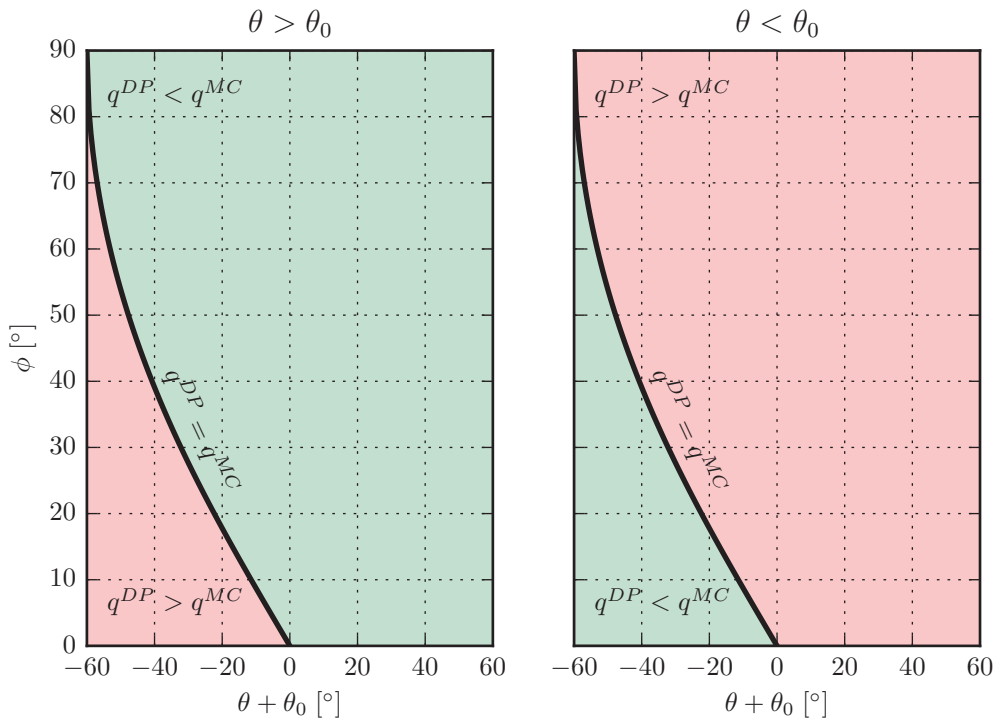
Figure 1: Variability of Drucker-Prager shear strength vs. Mohr-Coulomb shear strength with respect to Lode angles θ_0 and θ and the friction angle ϕ. The $q^{DP}=q^{MC}$ line represents the equivalent friction angles for the case of the DP criterion.

Clearly, the derived values of α and k depend on the Lode angle, so they are not constant for constants ϕ and c, but rely also on the intermediate principal stress σ_2.

4 Differences between MC and DP criteria

Let us consider now that α and k have been established for some fixed Lode angle, say θ_0, and inserted back to the DP criterion definition, namely Eq. (19). This will lead us, after some not very tedious algebraic transformations, to the following relation:

$$q^{DP} = Aq^{MC}, \tag{22}$$

where:

$$A = \frac{3\cos(\theta)}{\sqrt{3}\big(\sin(\theta) - \sin(\theta_0)\big)\sin(\phi) + 3\cos(\theta_0)}. \tag{23}$$

The DP shear strength can be then represented by the MC strength multiplied by a parameter A dependent on the friction angle ϕ and the following two Lode angles: θ_0, for which the DP parameters have been established, and θ, representing the current stress state.

Clearly if $A = 1$, then the DP and MC criteria are equivalent. This will occur in two cases: firstly if $\theta=\theta_0$ - which is quite obvious, and secondly if the following condition holds:

$$\sin\phi = \frac{\sqrt{3}(\cos\theta - \cos\theta_0)}{\sin\theta - \sin\theta_0} = \sqrt{3}\tan\left(-\frac{\theta + \theta_0}{2}\right). \tag{24}$$

One can note that Eq. (24) can be considered as a measure of equivalent friction angle (Griffiths 1990) for the DP condition when θ_0 and θ are given.

Further analysis of A shows that A>1, i.e. $q^{DP}>q^{MC}$, when one of the following occurs:

$$\theta > \theta_0 \quad \text{and} \quad \sin\phi < \sqrt{3}\tan\left(-\frac{\theta + \theta_0}{2}\right), \tag{25}$$

or

$$\theta < \theta_0 \quad \text{and} \quad \sin\phi > \sqrt{3}\tan\left(-\frac{\theta + \theta_0}{2}\right). \tag{26}$$

On the other hand if:

$$\theta > \theta_0 \quad \text{and} \quad \sin\phi > \sqrt{3}\tan\left(-\frac{\theta + \theta_0}{2}\right), \tag{27}$$

Table 1: Expressions for coefficient A and its minimum, maximum and average values for the selected values of θ_0 parameter.

θ_0 [°]	A	A_{min}	A_{max}	$\overline{A}$
-30	$\dfrac{2\sqrt{3}\cos\theta}{(2\sin\theta + 1)\sin\phi + 3}$	0.6	1.15	0.927
0	$\dfrac{3\cos\theta}{\sqrt{3}\sin\phi\sin\theta + 3}$	0.67	1.22	0.969
30	$\dfrac{2\sqrt{3}\cos\theta}{(2\sin\theta - 1)\sin\phi + 3}$	1.0	3.0	1.49
$-\arctan(\sin\phi/\sqrt{3})$	$\dfrac{\sqrt{3\sin^2\phi + 9}\cos\theta}{\left(\sqrt{\sin^2\phi + 3}\sin\theta + \sin\phi\right)\sin\phi + 3}$	0.6	1.0	0.897
4.22	$\dfrac{3\cos(\theta)}{\sqrt{3}(\sin(\theta) - 0.0736)\sin(\phi) + 2.99}$	0.7	1.3	1
-19.35	$\dfrac{3\cos(\theta)}{\sqrt{3}(\sin(\theta) + 0.331)\sin(\phi) + 2.83}$	0.61	1.06	0.909

or

$$\theta < \theta_0 \quad \text{and} \quad \sin\phi < \sqrt{3}\tan\left(-\frac{\theta + \theta_0}{2}\right), \quad (28)$$

then $A<1$ and $q^{DP}<q^{MC}$ are obtained. Additionally, considering the physical constraints on the friction angle, i.e. $\phi\in\langle0°, 90°\rangle$, it is observed that the inequality $\sin\phi > \sqrt{3}\tan(-(\theta + \theta_0)/2)$ is always true if only $\theta+\theta_0>0$. These variability considerations are presented compactly in Figure 1.

In computational practice some specific values of θ_0 are usually selected for fitting α and k parameters of the DP criterion with ϕ and c values. By setting θ_0 as -30°, 0 and 30°, the tensile, shear and compressive meridians, respectively, of the MC criterion are met. Furthermore, when the minimum of α coefficient is considered, then $\theta_0 = -\arctan(\sin\phi/\sqrt{3})$ is obtained and the DP yield surface inscribes the MC envelope. However, any other θ_0 can reasonably be used (see Section 5 for examples). Table 1 shows the expressions for A along with its minimum, maximum and average values for these typical θ_0 choices. Additionally, in Figure 2, the distribution of deviation of A from unity is presented. One can observe, that for $\theta_0 = 30°$ the Drucker-Prager shear strength is always greater (or equal) to the Mohr-Coulomb strength and it is exactly opposite for $\theta_0 = -\arctan(\sin\phi/\sqrt{3})$. For all other cases both possibilities are present, depending on θ and ϕ values. One can also verify that the overall maximum for A takes the value 3.0 and is achieved for $(\theta_0, \theta, \phi) = (30°, -30°, 90°)$, whereas the overall minimum is equal to 0.6 and is achieved for $(\theta_0, \theta, \phi) = (-30°, 30°, 90°)$. Thus, in general, the following inequality is valid:

$$0.6q^{MC} \leq q^{DP} \leq 3q^{MC}, \quad (29)$$

which means that the DP shear strength cannot be lower then 0.6 and greater than 3.0 times its MC counterpart.

5 Average difference between DP and MC vs. the DP parameters

An average value of the coefficient A in terms of θ_0 can be expressed as follows:

$$\overline{A}(\theta_0) = \frac{1}{\Omega}\int_\theta\int_\phi A(\theta_0, \theta, \phi)d\phi d\theta, \quad (30)$$

where $\Omega = [\theta] \times [\phi]$ is the chosen integration area. It would be of interest to look for such values of θ_0 for which $\overline{A} = 1$, i.e. the difference between MC and DP criteria vanishes, in this average sense. Another possibly interesting θ_0 point would be located at the minimum of $\overline{A}$.

Unfortunately the explicit formula for the integral defined by Eq. (30) seems to be impossible to be given, thus the integration has to be done numerically. Indeed, when A is integrated over the whole range of variability of θ and ϕ, i.e. for $\Omega=[-30°, 30°]\times[0°, 90°]$, then it reaches unity at $\theta_0=4.22°$ and
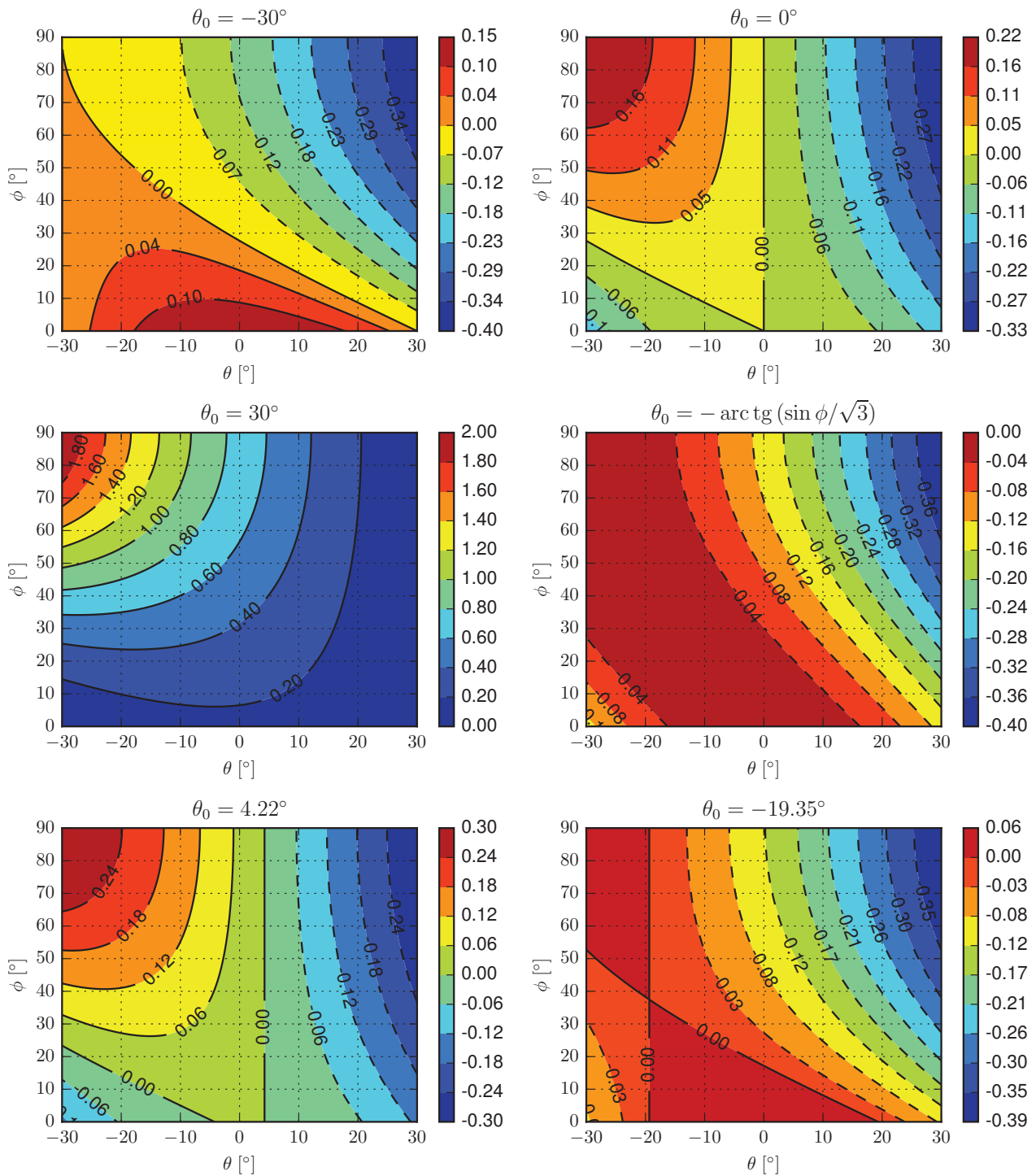
Figure 2: Deviation of coefficient *A* from unity for selected θ_0 values with respect to Lode angle θ and friction angle ϕ.

it attains its minimum at θ_0=-19.35° (see Figure 3). These θ_0 values can be treated as possible choices for establishing parameters of the DP criterion, especially when the best overall agreement with the MC shear strength is expected. See Table 1 and Figure 2 for more details about these θ_0 values.

However, one can also investigate other integration ranges for *A*, matching specific material properties and loading scenarios. For example in the case of natural sandy and gravelly soils the friction angle varies usually between 30° and 45°. If the compressive loading of such soil is assumed, i.e. $\theta \in \langle 0°, 30° \rangle$, then the condition $\overline{A} = 1$
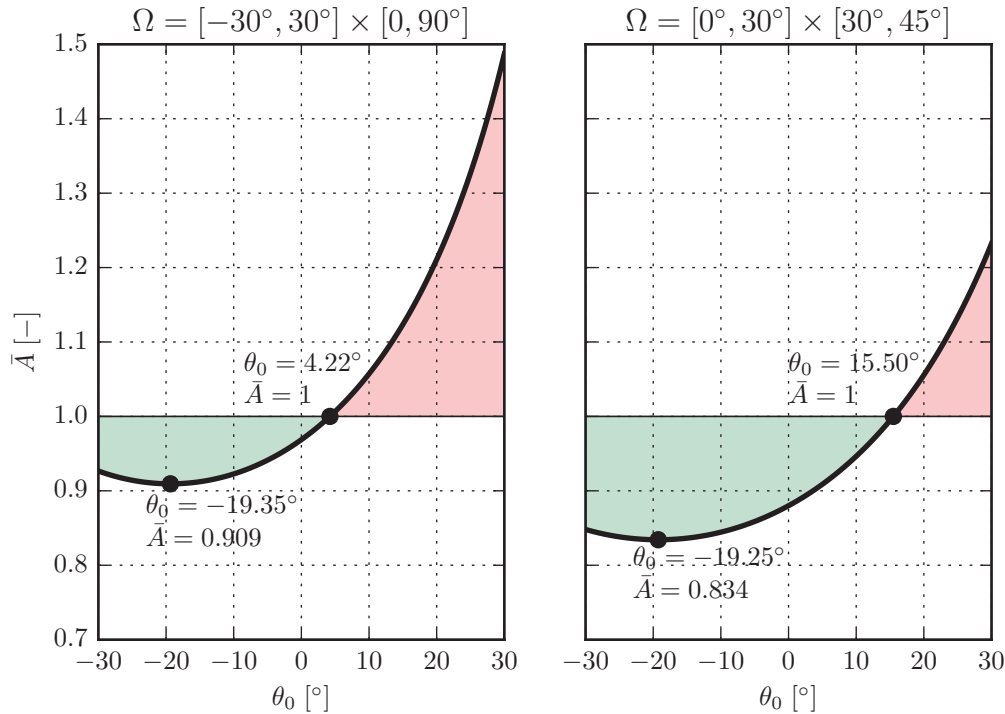
Figure 3: Integral $\overline{A}$ with respect to the θ_0 angle for different integration areas Ω.

will be obtained at θ_0=-15.5°, which is quite different from the previous result (see Figure 3).

This way the general method for obtaining the DP parameters that best fit the material behaviour and loading data is obtained.

6 A note on plane strain conditions

Elasto-plastic material models are often accompanied by plane strain conditions, which allow for dimension reduction from 3D to 2D. In this case, the *in-plane* principal stresses σ_1, σ_3 are used to compute the principal *out-of-plane* σ_2 stress via the following relation:

$$\sigma_2 = \nu(\sigma_1 + \sigma_3), \tag{31}$$

where ν is the Poisson ratio of the material. When the plastic flow occurs, it is very common to assume ν=0.5. Following the notation from Section 2 this choice corresponds to assuming the Lode angle θ=0° (or b=0.5 or a=0). In this specific case, the DP criterion become fully equivalent to the MC criterion, independently on the friction angle of the material, if only θ_0=0° is taken for fitting α and k parameters (see upper right graph in Figure 2). However, the validity of assuming the apparent Poisson ratio equal

to 0.5 is arguable [10, 11]. It should be emphasized that any other choice for ν makes the MC and DP equivalence disappear in plane strain conditions and the general 3D Drucker-Prager criterion has to be considered.

7 Summary and conclusions

In this paper the formula relating material shear strength predictions generated by the MC and the DP criteria is derived and analysed. This relation is of the form $q^{DP}=Aq^{MC}$, where A depends on the friction angle ϕ, Lode angle θ_0 for which the DP coefficients have been derived and Lode angle θ describing the current stress state. It should be also noted that A does not depend on the cohesion of the material. The variability considerations of this relation are summarized as follows:

–   the MC and DP criteria generate equivalent shear strengths if θ=θ_0 or $\sin\phi = \sqrt{3}\tan[-(\theta + \theta_0)/2]$,
–   for θ_0=30° the DP strength is always greater or equal to the MC strength and it is exactly opposite for $\theta_0 = -\arctan(\sin\phi/\sqrt{3})$,
–   the DP strength cannot be lower then 0.6 and greater than 3.0 times its Mohr-Coulomb counterpart, for the same friction angle and cohesion parameters,

- the new DP failure surfaces, minimizing the average discrepancy with the MC failure surface, can be proposed using the average value of A; for example for $\theta_0=4.22°$ and for $[\theta]\times[\phi]=[-30°,\ 30°]\times[0°,\ 90°]$ the average of A takes the value 1,
- in case of plane strain conditions the MC and DP criteria are equivalent, if the Poisson ratio in plastic zones is taken as 0.5 and the DP parameters are established with $\theta_0=0°$, otherwise this equivalence disappears.

The interval of variability of A value, i.e. $A\in\langle0.6,3\rangle$ can be considered as surprisingly wide. However the extreme values are obtained for friction angle $\phi=90°$, which is rather seldom observed, and for the maximum possible discrepancy between θ_0 and θ values (equal to 60°). Unfortunately, even for more realistic friction angles, i.e. for $\phi\leq45°$, the interval $A\in\langle0.68,\ 1.89\rangle$ is obtained, which is obviously narrower, but still significant. It seems that the key for achieving best overall agreement between the MC and DP predictions is the proper choice of θ_0 value. Indeed, quite good agreement is obtained for $\theta_0=0°$, where the maximum difference between θ_0 and θ is equal to 30° (see Figure 2). For this θ_0 choice and for $\phi\in\langle0°,\ 90°\rangle$ and $\theta\in\langle-30°,\ 30°\rangle$ the interval $A\in\langle0.67,\ 1.22\rangle$ is obtained. Furthermore, if we constrain the possible friction angles to $\phi\leq45°$ then this interval reduces to $A\in\langle0.72,\ 1.09\rangle$. Even better agreement, in the average sense, is possible to be achieved by means of the procedure described in Section 5, if only the variability ranges of the friction and Lode angles can be reasonably estimated for the problem under consideration. The value $\theta_0=4.22°$ is recommended for the most general case of ϕ and θ variability. For the reduced range of the friction angle, the value $\theta_0=7.92°$ can be used instead. It is generally discouraged to use in the computations the limiting values of Lode angle ($\theta_0=-30°$, $\theta_0=30°$) for fitting DP criterion parameters, unless it is really known that the considered loading conditions will be close to the triaxial extension or compression case.

The approach used in the paper may become a useful strategy for comparing also other shear strength criteria to the MC predictions. However, the linear relation of the explicit form given by Eqs. (22) and (23) might be simply impossible to be obtained in other cases and some more sophisticated functions should be investigated.

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