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# NON-EXTREME VARIANT OF THE SUPPORT MANAGEMENT MODEL OF A PRODUCTION-SUPPLY SYSTEM WITH THE STRUCTURAL PROCESS OF PRODUCT SUPPLY

The paper is dedicated to building a probabilistic analysis method of functioning a production-supply system with the structural process of product supply. This analysis is carried out in the non-extreme variant of warehouse filling level to which two independent streams of production (of the product) are directed by means of a transport subsystem. For this variant, four sets of equations that fulfilled the density function determining state probabilities of a multidimensional process characterizing operations of the system examination were derived.

**Keywords:** *production-supply system, process, product supply, non-extremal state, set of equations*

## 1. Introduction

Systems cooperating the production and supply systems are the object of research in various publications (e.g., [1–4, 9, 10, 12–18]). This article is a continuation of the research carried out in works [4, 6, 9, 10, 12, 16, 17], and especially in [5–7]. It is dedicated to building a new probabilistic model of the operation system in the non-extreme variant of warehouse filling level, taking into account the structural process of product

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delivery. Four sets of equations were introduced, which fulfil the density function probability in the case when the warehouse filling level does not reach limited levels.

## 2. Description of the system operation

To the recipient $E$ (e.g., power station), whose functioning is conditioned by the constant requirement $a$ of product units (e.g., Carbon), the stream of production results $y_1(t)$, $y_2(t)$ realized by the production subsystem $\tilde{P}$ is delivered in a continuous manner (e.g., by conveyor belts, pipelines, transmission lines). Random changes of the processes $y_1(t)$, $y_2(t)$ and unplanned breaks in work (failures) of the transport subsystem $\tilde{T}$ consisting of the subsystem $T_1$ and the subsystem $T_2$, are the factors decreasing the efficiency of the tested system. This efficiency can be increased while reducing the potential for interruptions in the supply of the adequate quantity of the product to the recipient $E$ by locating in the recipient's $E$ surroundings the warehouse-container $M$ with a specific volume $V$. The product streams $y_1(t)$, $y_2(t)$ are collected in the subsystem $M$, if the filling level $z(t)$ of the warehouse $M$ does not exceed $V$ and when $y_1(t) > a$ or $y_2(t) > a$. If the momentary value of the element $M$ is equal to $V$, $y_1(t) > a$ or $y_2(t) > a$, then the size of streams $y_1(t)$, $y_2(t)$ is limited to the level $a$. When the warehouse $M$ is empty and $y_1(t) < a$ and $y_2(t) < a$, then a situation unfavourable to the recipient $E$ arises. The probability determination of this event has practical meaning.

Cooperation of subsystems $\tilde{P}$, $\tilde{T}$, $M$ and $E$ is presented in Fig. 1.

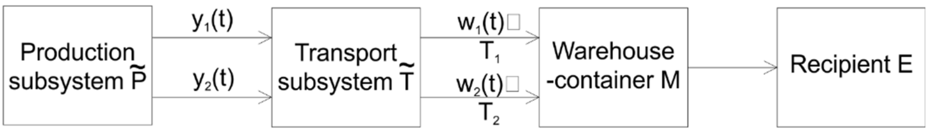


Fig. 1. Overall scheme of the tested system functioning

The inputs $w_1(t)$, $w_2(t)$ of the subsystem $M$ can be considered in two variants:

• the aggregated variant: $w_1(t) = y_1(t)$, $w_2(t) = y_2(t)$; processes $w_1(t)$, $w_2(t)$ together represent the production subsystem $\tilde{P}$ and the transport subsystem $\tilde{T}$ [5, 6],

• the structural variant: $w_1(t) = y_1(t)v_1(t)$, $w_2(t) = y_2(t)v_2(t)$; processes $w_1(t)$, $w_2(t)$ explicitly take into account both the production subsystem $\tilde{P}$ (product streams $y_1(t)$, $y_2(t)$), and the transport subsystem $\tilde{T}$ (processes $v_1(t)$, $v_2(t)$).

The processes $v_1(t)$, $v_2(t)$ characterizing operation of the subsystem $\tilde{T}$ are specified by the formula:

$$v_i(t) = \begin{cases} 1 \text{ when the subsystem } T_i \text{ is in the working state} \\ 0 \text{ when the subsystem } T_i \text{ is not working (is in breakdown state)} \end{cases} \quad \text{for } i = 1, 2$$

The process $v_1(t)$ describes the operation of the subsystem $T_1$, and the process $v_2(t)$ – operation of the subsystem $T_2$.

Let us denote by $\lambda_1$, $\lambda_2$, $\beta_1$, $\beta_2$ intensities of formation and disappearance of the subsystem $T_1$ breakdown (indicator 1) and the subsystem $T_2$ (indicator 2). We denote the states of the transport subsystem $\tilde{T}$ by $A_{11}$, $A_{10}$, $A_{01}$, $A_{00}$, where 1 indicates the working state, and 0 – the breakdown state (e.g., the state $A_{10}$ is the working state of the subsystem $T_1$ and the breakdown of the subsystem $T_2$). Allowable transitions between the states are shown in Fig. 2:

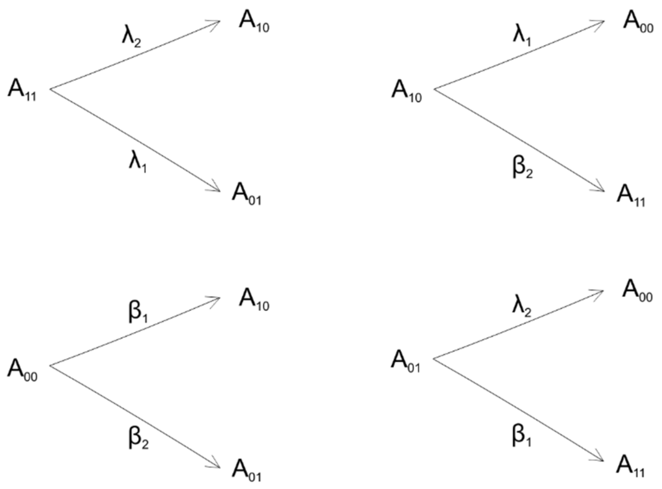


Fig. 2. Schemes of transitions between the states of the transport subsystem $\tilde{T}$

It is assumed that the operation process of the transport system $\tilde{T}$ is independent of the product streams of the production subsystem $\tilde{P}$.

## 3. Theoretical characteristics of the system operation

The functioning of the examined system is described by the processes ($y_1(t)$, $y_2(t)$, $v_1(t)$, $v_2(t)$, $z(t)$. We assume that the subsystems $T_1$ and $T_2$ operate independently, and the processes $y_1(t)$ and $y_2(t)$, controlling the warehouse filling level $z(t)$, are Markov's processes of a finite number of states. Let us denote states of the supply product stream $y_1(t)$ to the subsystem $M$ by $y_{11}, y_{12}, ..., y_{1n}$, and states of the product stream $y_2(t)$ by $y_{21}, y_{22}, ..., y_{2m}$.

Intensities of the transition between states (levels of product delivery) of the process $y_1(t)$ and $y_2(t)$ are denoted respectively by $\pi_{jk}^{(1)}$ and $\pi_{sk}^{(2)}$, as it is schematically written in the form:

$$y_{1j} \xrightarrow{\pi_{jk}^{(1)}} y_{1k} \quad \text{for } j \neq k \tag{1}$$

$$y_{2s} \xrightarrow{\pi_{si}^{(2)}} y_{2i} \quad \text{for } s \neq i \tag{2}$$

Receiving values supporting the management process of the examined system requires designation of the probabilities of the system states $P(y_1(t), y_2(t), v_1(t), v_2(t), z(t))$, and hence the likelihood that at a fixed point of time $t$ the supply stream of the product $y_1(t)$ will be at the state

$$y_1\colon y_{11}, y_{12}, ..., y_{1n} \tag{3}$$

the supply stream $y_2(t)$ will have the state

$$y_2\colon y_{21}, y_{22}, ..., y_{2m} \tag{4}$$

the process $v_1(t)$ describing the $T_1$ subsystem operation will be at the state

$$u_1\colon 1, 0 \tag{5}$$

the process $v_2(t)$ describing the $T_2$ subsystem operation will have the state

$$u_2\colon 1, 0, \tag{6}$$

and at the same time, the level of filling of the warehouse (container) will be $z$.

For each specific $z$, $0 < z < V$, this probability disappears:

$$P(y_1(t), y_2(t), v_1(t), v_2(t), z(t) = z) = 0$$

because there are uncountably many values of $z \in (0, V)$.

Therefore, the probability density function $f_{x_{2i},u_2}^{x_{1k},u_1}(z,t)$ is introduced, defined by the formula:

$$P(x_1(t)=x_{1k},v_1(t)=u_1,x_2(t)=x_{2i},v_2(t)$$
$$=u_2,a_1<z(t)<b_1)=\int_{a_1}^{b_1} f_{x_{2i},u_2}^{x_{1k},u_1}(z,t)dz \tag{7}$$

where $0 \le a_1 < b_1 \le V$, and $x_{1k}$ is an auxiliary introduced $k$-state of the process $x_1(t) = y_1(t) - a(x_{1k} = y_{1k} - a, k = 1, 2, ..., n)$, and $x_{2i}$ means the $i$-state of the process $x_2(t) = y_2(t) - a(x_{2i} = y_{2i} - a, i = 1, 2, ..., m)$.

The density functions $f_{x_{2i},u_2}^{x_{1k},u_1}(z, t)$ can be treated as a function sequence of two variables $z$, $t$ numbered respectively by the states $x_{1k}$, $u_1$, $x_{2i}$, $u_2$ of the processes $x_1(t)$, $v_1(t)$, $x_2(t)$, $v_2(t)$.

Analysis of the examined system functioning will be conducted in three variants, namely:

- Partial filling of the warehouse, $0 < z(t) < V$,
- Lower barrier $z(t) = 0$,
- Upper barrier, $z(t) = V$.

These cases should be considered individually because they correspond to different operating conditions of the system.

In order to obtain quantitative characteristics aimed at improving the efficiency of the system operation (Fig. 1), it is sufficient to determine the probabilities (7) and $P(x_1(t) = x_{1k}, v_1(t) = u_1, x_2(t) = x_{2i}, v_2(t) = u_2, z(t) = 0)$, $P(x_1(t) = x_{1k}, v_1(t) = u_1, x_2(t) = x_{2i}, v_2(t) = u_2, z(t) = V)$. Equation (7) expresses the probability that at a fixed time $t$ the inventory level in the warehouse $M$ belongs to the range $(a_1, b_1)$, but the stages of the processes $x_1(t)$, $v_1(t)$, $x_2(t)$, $v_2(t)$ are respectively $x_{1k}$, $u_1$, $x_{2i}$, $u_2$. The meaning of the other two probabilities is analogous. The probability $P(x_1(t) = x_{1k}, v_1(t) = u_1, x_2(t) = x_{2i}, v_2(t) = u_2, 0 < z(t) < V)$ we find from the formula

$$P\left(x_1(t)=x_{1k},\ v_1(t)=u_1,\ x_2(t)=x_{2i},\ v_2(t)=u_2,\ 0<z(t)<V\right)$$
$$=\lim_{\substack{a_1\to 0\\ b_1\to V}} P\left(x_1(t)=x_{1k},\ v_1(t)=u_1,\ x_2(t)=x_{2i},\ v_2(t)=u_2,\ a_1<z(t)<b_1\right)$$

To calculate the probability defined by Eq. (7), the method of determining the density functions $f_{x_{2i},1}^{x_{1k},1}(z,t)$, $f_{x_{2i},0}^{x_{1k},1}(z,t)$, $f_{x_{2i},1}^{x_{1k},0}(z,t)$, $f_{x_{2i},0}^{x_{1k},0}(z,t)$, and should be given.

## 4. Non-extreme state of the warehouse filling

Analysis of the examined system will be carried out in the first variant, that is, when the level of the warehouse $M$ inventory $z(t)$ satisfies the condition:

$$0 < z(t) < V \tag{8}$$

In this variant, the transport subsystems $T_1$ and $T_2$ can supplement the warehouse filling, and the recipient $E$ has his demand guaranteed.

Further, we derive equations which satisfy the above-mentioned four probability density functions. In order to obtain these equations, the formula for differentiable functions, known as Taylor's formula, will be used

$$h(z+\Delta z, t) = h(z,t) + \frac{\partial h(z,t)}{\partial z}\Delta z + o(\Delta z) \tag{9}$$

Where $o(\Delta z)$ means the infinitely small value of an order higher than $\Delta z$:

$$\lim_{\Delta z \to 0} \frac{o(\Delta z)}{\Delta z} = 0 \tag{9a}$$

Now, we will derive an equation that will fulfil the density function $f_{x_{2i},1}^{x_{1k},1}(z,t)$. Under the terms of the system operation in the variant (8), we have:

$$\begin{aligned}
f_{x_{2i'},1}^{x_{1k'},1}(z,t+\tau) &\approx f_{x_{2i},1}^{x_{1k},1}[z-(x_{1k}+x_{2i}+a)\tau, t][1-(\pi_k^{(1)}+\pi_i^{(2)}+\lambda)\tau] \\
&+\sum_{k'\neq k} f_{x_{2i},1}^{x_{1k'},1}[z-(x_{1k'}+x_{2i}+a)\tau, t]\pi_{k'k}^{(1)}\tau + \sum_{i'\neq i} f_{x_{2i'},1}^{x_{1k},1}[z-(x_{1k}+x_{2i'}+a)\tau, t]\pi_{i'i}^{(2)}\tau \\
&+\sum_{\substack{k'\neq k\\ i'\neq i}} f_{x_{2i'},1}^{x_{1k'},1}[z-(x_{1k'}+x_{2i'}+a)\tau, t]\pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau \\
&+f_{x_2,0}^{x_{1k},1}[z-x_{1k}\tau, t]\beta_2\tau + \sum_{k'\neq k} f_{x_{2i},0}^{x_{1k'},1}[z-x_{1k'}\tau, t]\beta_2\tau\pi_k^{(1)}\tau \\
&+\sum_{i'\neq i} f_{x_{2i'},0}^{x_{1k},1}[z-x_{1k}\tau, t]\beta_2\tau\pi_{i'i}^{(2)}\tau + \sum_{k'\neq k,\, i'\neq i} f_{x_{2ii'},0}^{x_{1k'},1}[z-x_{1k'}\tau, t]\pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau\beta_2\tau \\
&+f_{x_{2i},1}^{x_{1k},0}[z-x_{2i}\tau, t]\beta_1\tau + \sum_{k'\neq k} f_{x_{2i},1}^{x_{1k'},0}[z-x_{2i}\tau, t]\beta_1\tau\pi_{k'k}^{(1)}\tau \\
&+\sum_{i'\neq i} f_{x2i',1}^{x_{1k},0}[z-x_{2i'}\tau, t]\beta_1\tau\pi_{i'i}^{(2)}\tau + \sum_{k'\neq k,\, i'\neq i} f_{x_{2i'},1}^{x_{1k'},0}[z-x_{2i'}\tau, t]\pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau\beta_1\tau
\end{aligned} \tag{10}$$

where (see Figs. 1, 2)

$$\lambda = \beta_1 + \beta_2 \tag{11}$$

$$\pi_k^{(1)} = \sum_{l\neq k}\pi_{kl}^{(1)}, \quad \pi_i^{(2)} = \sum_{l\neq i}\pi_{il}^{(2)} \tag{12}$$

The meaning of the relation (10) is as follows: the first element in this equation informs about the probability of remaining in the state ($x_{1k,1}$, $x_{2i,1}$). This chance is 1 minus the sum of the output intensities from the states ($x_{1k,1}$, $x_{2i,1}$) (e.g., [8]). In our case, the output intensity from the state $x_{1k}$ is equal to $\pi_k^{(1)}$ (Fig. 1, Eq. (12)), from the state $x_{2i}$ is equal to $\pi_i^{(2)}$ (Fig. 2, Eq. (12)), and from the state (1, 1) is equal to $\lambda$ (Fig. 2, Eq. (11)). In this case, the fact that in "simple" processes the double state changes were of an order higher than $\tau$ was exploited. This is taken into account by asymptotic equality ≈, which means that the term $o(\tau)$ is omitted, that satisfies the condition

$$\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0 \tag{13}$$

Regardless of the state changes $x_{1k}$ and $x_{2i}$, the changes in the level of filling warehouse $z$ followed. They are controlled by both the process $y_1(t)$, as well as by the production stream $y_2(t)$. If at the time $t$ there was the state ($x_{1k}$, $x_{2i}$), then at the time $\tau$ the warehouse state was $(x_{1k} + x_{2i} + a)\tau$. Thus, when at the time $t + \tau$ the warehouse filling level was $z$, then at the moment $t$ it had to be $z - (x_{1k} + x_{2i} + a)\tau$. This very fact is taken into account in the first element of the Eq. (10). Similarly, the content of other elements is explained. Analogously to Eq. (10), the other three equations are derived:

$$\begin{aligned} & f_{x_{2i},0}^{x_{1k},1}(z, t+\tau) \approx f_{x_{2i},0}^{x_{1k},1}[z - x_{1k}\tau, t][1 - (\pi_k^{(1)} + \pi_i^{(2)} + (\lambda_1 + \beta_2))\tau] \\ & + f_{x_{2i},1}^{x_{1k},1}[z - (x_{1k} + x_{2i} + a)\tau, t]\lambda_2\tau + f_{x_{2i},0}^{x_{1k},0}[z + a\tau, t]\beta_1\tau \\ & + \sum_{k' \neq k} f_{x_{2i},0}^{x_{1k'},1}[z - x_{1k'}\tau, t]\pi_{k'k}^{(1)}\tau + \sum_{i' \neq i} f_{x_{2i'},0}^{x_{1k},1}[z - x_{1k}\tau, t]\pi_{i'i}^{(2)}\tau \end{aligned} \tag{14}$$

$$\begin{aligned} & f_{x_{2i},1}^{x_{1k},0}(z, t+\tau) \approx f_{x_{2i},1}^{x_{1k},0}[z - x_{2i}\tau, t][1 - (\pi_k^{(1)} + \pi_i^{(2)} + (\lambda_2 + \beta_1))\tau] \\ & + f_{x_{2i},1}^{x_{1k},1}[z - (x_{1k} + x_{2i} + a)\tau, t]\lambda_1\tau + f_{x_{2i},0}^{x_{1k},0}[z + a\tau, t]\beta_2\tau \\ & \sum_{k' \neq k} f_{x_{2i},1}^{x_{1k'},0}[z - x_{2i}\tau, t]\pi_{k'k}^{(1)}\tau + \sum_{i' \neq i} f_{x_{2i'},1}^{x_{1k},0}[z - x_{2i'}\tau, t]\pi_{i'i}^{(2)}\tau \end{aligned} \tag{15}$$

$$\begin{aligned} & f_{x_{2i},0}^{x_{1k},0}(z, t+\tau) \approx f_{x_{2i},0}^{x_{1k},0}[z + \alpha\tau, t][1 - (\beta + \pi_k^{(1)} + \pi_i^{(2)})\tau] \\ & + f_{x_{2i},0}^{x_{1k},1}[z - x_{1k}\tau, t]\lambda_1\tau + f_{x_{2i},1}^{x_{1k},0}[z - x_{2i}\tau, t]\lambda_2\tau \\ & + \sum_{k' \neq k} f_{x_{2i},0}^{x_{1k'},0}(z + \alpha\tau, t)\pi_{k'k}^{(1)}\tau + \sum_{i' \neq i} f_{x_{2i'},0}^{x_{1k},0}(z + \alpha\tau, t)\pi_{i'i}^{(2)}\tau \end{aligned} \tag{16}$$

where $\beta = \beta_1 + \beta_2$. We will transform Eq. (10) using Taylor's formula (9):

$$
\begin{aligned}
& f_{x_{2i},1}^{x_{1k},1}(z,t+\tau) \\
& \approx \left\{ f_{x_{2i}}^{x_{1k}}(z,t) + \frac{\partial f_{x_{2i}}^{x_{1k}}(z,t)}{\partial z}[-(x_{1k}+x_{2i}+a)\tau] + o[-(x_{1k}+x_{2i}+a)\tau] \right\} \\
& \times \left[ 1-(\pi_k^{(1)}+\pi_i^{(2)})\tau \right] \\
& + \sum_{k'\neq k} \left\{ f_{x_{2i},1}^{x_{1k},1}(z,t) + \frac{\partial f_{x_{2i}}^{x_{1k}}(z,t)}{\partial z}[-(x_{1k}+x_{2i}+a)\tau] + o[-(x_{1k}+x_{2i}+a)\tau] \right\} \pi_{k'k}^{(1)}\tau \\
& + \sum_{i'\neq i} \left\{ f_{x_{2i'},1}^{x_{1k},1}(z,t) + \frac{\partial f_{x_{2i'},1}^{x_{1k},1}(z,t)}{\partial z}[-(x_{1k}+x_{2i'}+a)\tau] + o[-(x_{1k}+x_{2i'}+a)\tau] \right\} \pi_{k'k}^{(1)}\tau \\
& + \sum_{\substack{k'\neq k \\ i'\neq i}} \left\{ f_{x_{2i'},1}^{x_{1k'},1}(z,t) + \frac{\partial f_{x_{2i'},1}^{x_{1k'},1}(z,t)}{\partial z}[-(x_{1k'}+x_{2i'i'}+a)\tau] + o[-(x_{1k'}+x_{2i'}+a)\tau] \right\} \pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau \\
& + \left\{ f_{x_{2i},0}^{x_{1k},1}(z,t) + \frac{\partial f_{x_{2i},0}^{x_{1k},1}(z,t)}{\partial z}[-x_{1k}\tau] + o[-x_{1k}\tau] \right\} \beta_2\tau \\
& + \sum_{k'\neq k} \left\{ f_{x_{2i},0}^{x_{1k'},1}(z,t) + \frac{\partial f_{x_{2i},0}^{x_{1k'},1}(z,t)}{\partial z}[-x_{1k'}\tau] + o[-x_{1k'}\tau] \right\} \beta_2\tau\pi_{k'k}^{(1)}\tau \\
& + \sum_{i'\neq i} \left\{ f_{x_{2i'},0}^{x_{1k},1}(z,t) + \frac{\partial f_{x_{2i'},0}^{x_{1k},1}(z,t)}{\partial z}[-x_{1k}\tau] + o[-x_{1k}\tau] \right\} \beta_2\tau\pi_{i'i}^{(2)}\tau \\
& + \sum_{k'\neq k, i'\neq i} \left\{ f_{x_{2i'}}^{x_{1k'}}(z,t) + \frac{\partial f_{x_{2i}'}^{x_{1k}'}(z,t)}{\partial z}[-x_{1k'}\tau] + o[-x_{1k'}\tau] \right\} \pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau\beta_2\tau \\
& + \left\{ f_{x_{2i},1}^{x_{1k},0}(z,t) + \frac{\partial f_{x_{2i},1}^{x_{1k},0}(z,t)}{\partial z}[-x_{2i}\tau] + o[-x_{2i}\tau] \right\} \beta_1\tau \\
& + \sum_{k'\neq k} \left\{ f_{x_{2i},1}^{x_{1k'},0}(z,t) + \frac{\partial f_{x_{2i},1}^{x_{1k'},0}(z,t)}{\partial z}[-x_{2i}\tau] \right\} \beta_1\tau\pi_{k'k}^{(1)}\tau \\
& + \sum_{i'\neq i} \left\{ f_{x_{2i'},1}^{x_{1k},0}(z,t) + \frac{\partial f_{x_{2i'},1}^{x_{1k},0}(z,t)}{\partial z}[-x_{2i'}\tau] + o[-x_{2i'}\tau] \right\} \beta_1\tau\pi_{i'i}^{(2)}\tau \\
& + \sum_{k'\neq k, i'\neq i} \left\{ f_{x_{2i'},1}^{x_{1k'},0}(z,t) + \frac{\partial f_{x_{2i'},1}^{x_{1k'},0}(z,t)}{\partial z}[-x_{2i'}\tau] + o[-x_{2i'}\tau] \right\} \pi_{k'k}^{(1)}\tau\pi_{i'i}^{(2)}\tau\beta_1\tau \qquad (17)
\end{aligned}
$$

We shall now apply successively the following operations to Eq. (17):

- let us move the function $f_{x_{2i},1}^{x_{1k},1}(z,t)$ to the left side of Eq. (17),
- let us divide both sides of the resulting formula by $\tau$,
- let us move on both sides to the border with $\tau \to 0$.

As a result of these operations, the asymptotic equation (17) passes into (18) on the basis of the formulas (13) and (9a). Equation (18) presents equations which satisfy the density function $f_{x_{2i},1}^{x_{1k},1}(z,t)$ specifying the probabilities expressed by the Eq. (7).

$$\frac{\partial f_{x_{2i},1}^{x_{1k},1}(z,t)}{\partial t} = -\frac{\partial f_{x_{2i},1}^{x_{1k},1}(z,t)}{\partial z}(x_{1k}+x_{2i}+a) - f_{x_{2i},1}^{x_{1k},1}(z,t)\left[\pi_k^{(1)}+\pi_i^{(2)}+\lambda\right]$$
$$+\sum_{k'\neq k} f_{x_{2i},1}^{x_{1k'},1}(z,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'},1}^{x_{1k},1}(z,t)\pi_{i'i}^{(2)} + f_{x_{2i},0}^{x_{1k},1}(z,t)\beta_2 + f_{x_{2i},1}^{x_{1k},0}(z,t)\beta_1 \tag{18}$$

for $0 < z < V$, $k = 1, 2, ..., n$, $i = 1, 2, ..., m$.

Using Taylor's formula to the Eqs. (14)–(16) similarly to Eq. (17), then three operations given earlier, we obtain three equations that fulfil the other density functions $f_{x_{2i},0}^{x_{1k},1}(z,t)$, $f_{x_{2i},1}^{x_{1k},0}(z,t)$, and $f_{x_{2i},0}^{x_{1k},0}(z,t)$.

$$\frac{\partial f_{x_{2i},0}^{x_{1k},1}(z,t)}{\partial t} = -\frac{\partial f_{x_{2i},0}^{x_{1k},1}(z,t)}{\partial z}x_{1k} - f_{x_{2i},0}^{x_{1k},1}(z,t)\left[\pi_k^{(1)}+\pi_i^{(2)}+(\lambda_1+\beta_2)\right]$$
$$+f_{x_{2i},1}^{x_{1k},1}(z,t)\lambda_2 + f_{x_{2i},0}^{x_{1k},0}(z,t)\beta_1 + \sum_{k'\neq k} f_{x_{2i},0}^{x_{1k'},1}(z,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'},0}^{x_{1k},1}(z,t)\pi_{i'i}^{(2)} \tag{19}$$

for $0 < z < V$, $k = 1, 2, ..., n$, $i = 1, 2, ..., m$.

$$\frac{\partial f_{x_{2i},1}^{x_{1k},0}(z,t)}{\partial t} = -\frac{\partial f_{x_{2i},1}^{x_{1k},0}(z,t)}{\partial z}x_{1k} - f_{x_{2i},1}^{x_{1k},0}(z,t)\left[\pi_k^{(1)}+\pi_i^{(2)}+(\lambda_2+\beta_1)\right]$$
$$+f_{x_{2i},1}^{x_{1k},1}(z,t)\lambda_1 + f_{x_{2i},0}^{x_{1k},0}(z,t)\beta_2 + \sum_{k'\neq k} f_{x_{2i},1}^{x_{1k'},0}(z,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'},0}^{x_{1k},0}(z,t)\pi_{i'i}^{(2)} \tag{20}$$

for $0 < z < V$, $k = 1, 2, ..., n$, $i = 1, 2, ..., m$.

$$\frac{\partial f_{x_{2i},0}^{x_{1k},0}(z,t)}{\partial t} = -f_{x_{2i},0}^{x_{1k},0}(z,t)\left[\pi_k^{(1)}+\pi_i^{(2)}+\beta\right] + \frac{\partial f_{x_{2i},0}^{x_{1k},0}(z,t)}{\partial z}a$$
$$+f_{x_{2i},0}^{x_{1k},1}(z,t)\lambda_1 + f_{x_{2i},0}^{x_{1k},0}(z,t)\lambda_2 + \sum_{k'\neq k} f_{x_{2i},0}^{x_{1k'},0}(z,t)\pi_{k'k}^{(1)} + \sum_{i'\neq i} f_{x_{2i'},0}^{x_{1k},0}(z,t)\pi_{i'i}^{(2)} \tag{21}$$

for $0 < z < V$, $k = 1, 2, ..., n$, $i = 1, 2, ..., m$.

## 5. Summary

Taking into account the structural process of the product supply when, in the explicit way, both the production subsystem and the transport subsystem have an impact on the state of filling the warehouse, four equations that satisfied the probability density functions were derived. These equations constitute the probabilistic description of the examined system in the case when this state does not reach the limit values. In the next research work, a system operation analysis in other variants of its operation will be presented. The result expressed by the relations (18)–(21), together with the probabilistic description of the examined system work in the case of both barriers, will create opportunity to obtain characteristics of the system enabling to increase efficiency of its operations.

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