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Applications of Mathematics and Statistics in Economics

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BOND VALUATION AND INTEREST RATE MODELS USING *MATHEMATICA*

1. Bond valuation

We start with the bond pricing. The theoretical bond price can be computed as the present value of the cash flows obtained by the bond owner. In general the price of coupon bond with continuous compounding is defined as

$$P = \sum_{i=1}^{n} C\delta \cdot e^{-R(T_0,T_i)(T_i - T_0)} + F \cdot e^{-R(T_0,T_n)(T_n - T_0)},$$

where P denotes the bond price, C is the annual coupon, δ is the interval of coupon payments, F denotes the face value of bond and $R(T_0,T_i)$ is the interest rate for the time period from T_0 to T_i. In *Mathematica* we can write the procedure as

```
In[1]:= n=4; Array[R,n];
In[2]:= R={0.05,0.06,0.07,0.08};
In[3]:= Bondprice[F_,coupon_,T_,delta_]:=
        Sum[coupon*delta*Exp[-R[[i]]*(i*delta)],{i,1,n}]
        +F*Exp[-R[[n]]*T];
In[4]:= Bondprice[100,10,1.5,0.5]
Out[4]= 107.04
```

For selected interest rates and bond values (face value F = 100\$, semiannual coupon C = 5%, maturity T = 1.5) we calculate the bond price P = 107.04\$.

We now discuss how we can calculate interest rates from the coupon bond prices. This procedure is known as the bootstrap method. To illustrate it we consider 5 coupon bonds with known parameters – face value, maturity, coupon and bond price. Using the expression above we substitute particular parameters and we calculate the interest rate for individual maturities. The data and results are in Table 1. The face value of each bond is 100$, the interval of coupon payment is 6 months.

Table 1. Data for Bootstrap Method

Maturity (years)	Annual coupon (%)	Bond price ($)	Interest rate (%)
0.25	0	97.7	9.31
0.5	0	95.3	9.63
1	0	90.7	9.76
1.5	8	97.1	9.89
2	10	99.4	10.10

For example, the 1.5-year interest rate denoted by $R_{1.5}$ is calculated from equation

$$97.1 = 4e^{-0.0963\cdot 0.5} + 4e^{-0.0976\cdot 1} + 104e^{-R_{1.5}\cdot 1.5}$$
$$R_{1.5} = 9.89\%.$$

In *Mathematica* the equation can be solved by

```
In[1]:= FindRoot[97.1==4*E^(-0.0963*0.5)
        +4*E^(-0.09761*1)+104*E^(-R4*1.5),{R4,0.095}]
Out[1]= {R4->0.0989106}
```

Figure 1 shows the interest rates given by the bootstrap method.

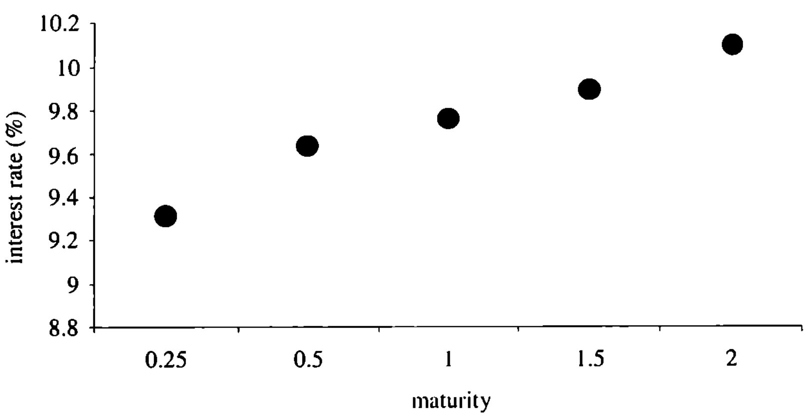

Fig. 1. Data for Bootstrap Method

2. Duration

The duration (denoted by D) of a bond is a measure of how long (on average) the owner of the bond has to wait before receiving cash payments. We define it as

$$D = \frac{\sum_{i=1}^{n} t_i X_i e^{-y \cdot t_i}}{\sum_{i=1}^{n} X_i e^{-y \cdot t_i}},$$

where X_i denotes the payment at time t_i, y is the yield. In *Mathematica* the algorithm for calculation of the duration is

```
In[1]:= n=6; time={0.5,1,1.5,2,2.5,3}; coupon=4; F=100;
        y=0.1;
In[2]:= PV=Sum[coupon*E^(-y*time[[i]]),{i,1,n}]
        +F*E^(-y*time[[n]])
Out[2]= 94.3023
In[3]:= duration=(Sum[time[[i]]*coupon*E^(-y*time[[i]]),
        {i,1,n}]
        +time[[n]]*F*E^(-y*time[[n]]))/PV
Out[3]= 2.71636
```

So for the bond data: yield = 10%, face value F = 100$, semiannual coupon = 4% and maturity 3 years we calculate the duration 2.72.

3. Monte-Carlo simulation of the Cox-Ingersoll-Ross spot interest rate process

The risk-neutral process for r in the Cox-Ingersoll-Ross (CIR) model is

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW,$$

where κ, θ and σ are constants and W is Brownian motion. θ is the long-run mean interest rate. Set $\kappa = 3\%$, $\theta = 5\%$, $\sigma = 0.15$ and $\Delta T = 0.0002$ and $r_0 = 4\%$. In the software *Mathematica* we can write the algorithm simulating the Cox--Ingersoll-Ross model as

```
In[1]:= <<Statistics`NormalDistribution`
In[2]:= Z[mu_,sigma_]:=
        Random[NormalDistribution[mu,sigma]];
```

```
In[3]:= n=250; Array[r,n];
In[4]:= r[0]=0.04; kappa=0.03; theta=0.05; deltaT=0.0002;
        sigma=0.15;
In[5]:= simCIR=For[i=0,i<=n,i++,
                 r[i+1]=r[i]+kappa*(theta-r[i])*deltaT
                 +sigma*Sqrt[r[i]]*Z[0,1]*Sqrt[deltaT]]
In[6]:= simCIR=Table[r[i],{i,0,n}]
In[7]:= ListPlot[simCIR,PlotJoined->True]
```

Fig. 2 shows the Monte-Carlo simulation of the CIR model (the number of steps n=250).

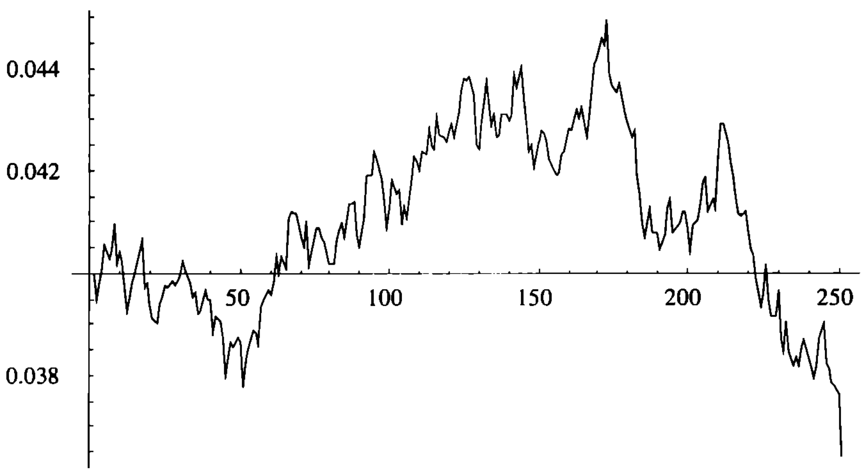

Fig. 2. Simulation of the CIR model

4. Summary

We introduced some algorithms in *Mathematica*, which can be useful in financial calculations. The procedures and illustrative examples for bond price calculation, the bootstrap method, the CIR simulation and duration are presented.

References

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WYCENA OBLIGACJI I MODELE STOPY PROCENTOWEJ WYKONANE ZA POMOCĄ PROGRAMU *MATHEMATICA*

Streszczenie

W artykule są dyskutowane zasady wyceny obligacji, modelowania stopy procentowej i czasu trwania. Najpierw wprowadzono podstawy obliczania ceny obligacji i zademonstrowano metody bootstrapowe w celu wyznaczenia stóp procentowych. Zaprezentowano także symulację modelu Coxa-Ingersolla-Rossa. Głównym celem artykułu jest demonstracja algorytmu prostych obliczeń z oprogramowaniem *Mathematica*.

Słowa kluczowe: obligacja, metody bootstrapowe, model Coxa-Ingersolla-Rossa, czas trwania, stopa procentowa.