Poisson integrals, Riesz transforms, and conjugacy for Laguerre expansions

Doctoral dissertation

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Contents

Preface

Classical harmonic analysis - the theory of Fourier series and Fourier integrals - underwent rapid development, stimulated by physical problems, in the eighteenth and nineteenth centuries; Dirichlet, Riemann, Lebesgue, Plancherel, Fejér, and F. Riesz formulated harmonic analysis as an independent mathematical discipline. Since then, in the twenteenth century, it separated into a multitude of significant branches, whose further development resulted in many applications in various fields of mathematics, including functional analysis, complex analysis, probability theory, and differential equations. This dissertation concerns the classical flow of the theory, which is known as nontrigonometric Fourier analysis. More precisely, we shall deal with Fourier series in certain special orthogonal functions and study the associated fundamental objects of harmonic analysis: maximal functions, heat-diffusion and Poisson integrals, Riesz transforms, and conjugate Poisson integrals. Systematic study of the aforementioned objects in the context of nontrigonometric orthogonal expansions was initiated in 1965 in the fundamental work [**MuS**] of Muckenhoupt and Stein who, in analogy to ordinary Fourier series, considered Gegenbauer (ultraspherical) expansions. They gave appropriate definitions of Poisson and alternate Poisson integrals, conjugate function, and conjugate Poisson integrals (in our terminology heat-diffusion and Poisson integrals, Riesz transform, conjugate Poisson integrals, respectively) in one dimension and obtained various results related to these operators, inter alia boundary behavior and L^p mapping properties. Then, according to Stein's suggestions, Muckenhoupt elaborated necessary tools and proved similar results for Hermite and Laguerre polynomial expansions [**Mu1, Mu2, Mu3**]. However, he worked in the unidimensional setting, and used methods which seem to be inapplicable in higher dimensions. But despite restricting to one dimension, the corresponding analysis is detailed and rather hard, particularly in the case of Laguerre expansions [**Mu3**].

Later, the work of Muckenhoupt was continued by many authors, who considered manifold types of expansions based on both Hermite and Laguerre polynomials and functions. There were several directions of research: to investigate multi-dimensional settings, to study higher order Riesz operators, and to consider expansions with respect to different systems of (Hermite and Laguerre) functions, which, in some sense, behave much better than polynomial expansions. In the 1980's and 1990's, many valuable results were obtained in this area, see [**Di, FGS, FoSc, GoSt, Gu, GST, Me, Pi, Sj1, Sj2, St, Th1, Th2, Th3**]. Furthermore, one can observe a growing interest in the harmonic analysis of classical orthogonal expansions in recent years, which manifested in the attention of many mathematicians such as J. García-Cuerva, C.E. Gutiérrez, P. Sjögren, S. Thangavelu, or J.L. Torrea, and numerous publications, for instance [**FSU1, FSU2, GIT, GMMST1, GMMST2, GMST1, GMST2, HRST, KeTh, MPS1, MPS2, PeSo, StTo**]. Most of them, however, concern Hermite expansions. Consequently, not so much is known about the Laguerre counterpart of the theory, which is essentially more involved, but on the other hand, richer and hence more interesting. Nevertheless, research in this direction faces many technical difficulties, seems to require a new approach, and therefore is substantially retarded in comparison with research on Hermite expansions.

The main purpose of this thesis is to contribute to the theory of Laguerre polynomial and function expansions both in its results and methods, which, as the author believes, enable further development of the subject and shed some new light on the interplay between Hermite and Laguerre expansions.

The structure of the dissertation is as follows.

In Chapter 1 we consider multi-dimensional expansions with respect to special Hermite functions and two different systems of Laguerre functions. We define corresponding heat-diffusion and Poisson integrals in a weighted L^p setting with weights from Muckenhoupt's A_p class, $1 \leq p < \infty$. Then their smoothness, boundary behavior and mapping properties are investigated. Heat-diffusion and Poisson integrals for Laguerre polynomial expansions were first studied by Muckenhoupt [**Mu1**]. Then Stempak [**St**], motivated by Muckenhoupt's paper, considered one-dimensional Laguerre expansions with respect to three different systems of Laguerre functions. Multi-dimensional Hermite function expansions in a weighted L^p setting have recently been considered by Stempak and Torrea [**StTo**]. Noteworthy, some aspects of weighted L^p theory for special Hermite and certain one-dimensional Laguerre expansions have also recently been treated by Kerman and Thangavelu [**KeTh**].

Chapter 2 is devoted to Riesz transforms associated with multi-dimensional polynomial Laguerre expansions of type α. As the principal result we prove that for $1 < p < \infty$ Riesz-Laguerre transforms are bounded operators in L^p equipped with the appropriate measure. What is also important and interesting, the corresponding L^p constants are independent of the dimension and the type multi-index α. Subsequently, we obtain boundedness and convergence results for the associated conjugate Poisson integrals. The main proof, motivated by the paper [**Gu**], is based on a suitable adaptation of the Littlewood-Paley-Stein theory [**S1**]. As a by-product we obtain L^p boundedness for the corresponding g-functions, results which are of interest in themselves. Riesz transforms and conjugate Poisson integrals for Laguerre expansions were first studied by Muckenhoupt [**Mu3**]. Recently, a g-function and Riesz transforms associated with the multi-dimensional Laguerre semigroup have been investigated by Gutiérrez, Incognito and Torrea, [**GIT**]. The technique of transference exploited there allowed to obtain $L^p, 1 < p < \infty$, boundedness results only for the discrete set of half-integer multi-indices α. We remove this restriction and consider all intermediate multi-indices.

Throughout Chapter 3, starting by giving suitable definitions, we focus on higher order Riesz transforms and related operators for multi-dimensional polynomial Laguerre expansions. The main results, boundedness in L^p, $1 < p < \infty$, of these transforms and weak type 1–1 of Riesz-Laguerre transforms of order 2, are obtained by means of transference from a Hermite setting, after restricting to half-integer multi-indices α. The corresponding L^p constants depend neither on the dimension nor on the type multi-index α. The method of transference was used by Dinger [**Di**], and recently has been developed by Gutiérrez et al., [**GIT**]. We provide a significant extension of this technique and show how to transfer higher order Riesz type operators and certain differential operators.

It is noteworthy that the results of Chapters 2 and 3 already found interesting applications in studying Sobolev spaces associated with polynomial Laguerre expansions [**GLLNU**]. Moreover, some techniques indirectly developed in Chapter 3 allowed to discover closed bilinear composition formulas for multi-dimensional Hermite and Gould-Hopper polynomials [**GrNo**].

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Basic notation

$$\begin{aligned}
\mathbb{N} &\equiv \text{the set of natural numbers, including } 0,\\
\mathbb{R} &\equiv \text{the set of real numbers},\\
\mathbb{C} &\equiv \text{the set of complex numbers},\\
\mathbb{R}_+ &= (0,\infty).
\end{aligned}$$

Let $d \in \mathbb{N}\backslash\{0\}$. Given $x \in \mathbb{R}^d$ or $x \in \mathbb{C}^d$, we denote its Euclidean norm by $|x|$ or $|x|_{\ell^2}$. For a multi-index $m = (m_1, \dots, m_d)$ we always understand $|m|$ as its length, i.e. $|m| = \sum_1^d m_i$. If $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, then

$$x^m \equiv x_1^{m_1} \cdots x_d^{m_d}.$$

The function classes

$$C_c^k,\ C_0^k,\ C_B^k,\ C^k,\ L^p,\ L^p(d\mu),\ L_{\text{loc}}^p(d\mu),$$

are defined in a standard manner and will consist of functions defined on $\mathbb{C}^d, \mathbb{R}^d$ or $\mathbb{R}_+^d$. For $1 \le p < \infty$ we denote the class of Muckenhoupt's weights with respect to the Lebesgue measure dx by $A_p = A_p(\Omega)$, the underlying space Ω being dependent on the context. The conjugate p' of p is defined by the identity $1/p + 1/p' = 1$.

Given $\alpha > -1$, one-dimensional *Laguerre polynomials* of type α are of the form

$$L_k^\alpha(x) = \frac{1}{k!} e^x x^{-\alpha} \frac{d^k}{dx^k}(e^{-x} x^{k+\alpha}), \qquad k \in \mathbb{N},\ x > 0.$$

Note, that each L_k^α is a polynomial of degree k. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha \in (-1,\infty)^d$, d-dimensional Laguerre polynomials of type α are tensor products of one-dimensional Laguerre polynomials, that is to say

$$L_k^\alpha(x) = \bigotimes_{i=1}^d L_{k_i}^{\alpha_i}(x_i), \qquad k \in \mathbb{N}^d,\ x \in \mathbb{R}_+^d.$$

Similarly, multi-dimensional *Hermite polynomials* H_k, $k \in \mathbb{N}^d$, are tensor products of the one-dimensional Hermite polynomials defined by

$$H_k(x) = e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \qquad k \in \mathbb{N},\ x \in \mathbb{R}.$$

Hermite functions h_k in $\mathbb{R}^d$ are of the form

$$h_k(x) = c_k H_k(x) e^{-|x|^2/2}, \qquad k \in \mathbb{N}^d,\ x \in \mathbb{R}^d,$$

where $c_k = \prod_{i=1}^d (\sqrt{\pi} 2^{k_i} \Gamma(k_i + 1))^{-1/2}$ is the normalizing factor.

We adopt the convention that constants may change their value from one use to the next. The notation $c = c_{\beta,\gamma,\dots}$ means that c is a constant depending *only* on $\beta, \gamma, \dots$. Constants are always strictly positive and finite.

CHAPTER 1

Heat-diffusion and Poisson integrals for special Hermite and Laguerre function expansions on weighted L^p spaces

1.1. Introduction

Heat-diffusion and Poisson integrals for Laguerre polynomial expansions were first studied by Muckenhoupt [**Mu1**]. Then Stempak [**St**], motivated by Muckenhoupt's paper, considered one-dimensional Laguerre expansions with respect to different systems of Laguerre functions. Multi-dimensional Hermite function expansions in weighted L^p setting have recently been considered by Stempak and Torrea [**StTo**]. Our aim is to go further and discuss multi-dimensional Laguerre and special Hermite function expansions in weighted L^p setting. We note that some aspects of weighted L^p theory for special Hermite and certain one-dimensional Laguerre expansions have been recently treated also by Kerman and Thangavelu [**KeTh**].

Here we consider expansions with respect to special Hermite functions and two different systems of Laguerre functions. We define pointwise corresponding heat-diffusion and Poisson integrals in weighted L^p setting with weights from Muckenhoupt's A_p class, $1 \leq p < \infty$. Then we investigate their smoothness, boundary behavior and mapping properties. In particular, we show that the associated maximal operators are dominated, up to a constant, by the Hardy-Littlewood maximal function or by the strong maximal function. We follow closely the technique used in [**StTo**] for ordinary multi-dimensional Hermite function expansions and the corresponding integrals.

This chapter is organized as follows. In Sections 1.2, 1.3 and 1.4 we treat in order special Hermite expansions, Laguerre expansions based on the system $\{\ell_k^\alpha\}$, and Laguerre expansions with respect to the system $\{\varphi_k^\alpha\}$. Main results of these sections are contained in Theorems 1.2.10, 1.3.6 and 1.4.5. Finally, in Section 1.5 we give some remarks on the connection between the Hermite and Laguerre cases, including an extension of the transference studied in [**Di**] and [**GIT**].

1.2. Special Hermite expansions

In this section we study heat-diffusion and Poisson semigroups associated with the special Hermite operator $\widetilde{\Delta}$, usually called the *twisted Laplacian*. The operator $\widetilde{\Delta}$ is closely related to the sublaplacian on the Heisenberg group $\mathbb{H}^n$, and special Hermite expansions play an important role in a better understanding of some problems on $\mathbb{H}^n$ (see [**Th4**]).

Let $n \geq 1$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{C}^n$. Then we have

$$\widetilde{\Delta} = -\Delta_x - \Delta_y + \frac{1}{4}(|x|^2 + |y|^2) - i \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right),$$

where $\Delta_x = \sum_1^n \partial_{x_j}^2$, $\Delta_y = \sum_1^n \partial_{y_j}^2$ are the standard Laplacians on $\mathbb{R}^n$. The set of eigenfunctions of this operator contains special Hermite functions $\Phi_{\alpha,\beta}$ $(\alpha, \beta \in \mathbb{N}^n)$. These form a complete orthonormal system in $L^2(\mathbb{C}^n)$ and are given by the following Fourier-Wigner transform of the usual Hermite functions h_α and h_β :

$$\Phi_{\alpha,\beta}(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} h_\alpha\left(\xi + \frac{y}{2}\right) h_\beta\left(\xi - \frac{y}{2}\right) d\xi, \quad z = x + iy \in \mathbb{C}^n.$$

The spectrum of $\widetilde{\Delta}$ is discrete and equals $\{2k + n : k \in \mathbb{N}\}$. Since we have $\widetilde{\Delta}\Phi_{\alpha,\beta} = (2|\beta| + n)\Phi_{\alpha,\beta}$, the eigenspace corresponding to the eigenvalue $2k + n$ is infinite-dimensional and spanned by $\{\Phi_{\alpha,\beta} : |\beta| = k\}$. For a function $f \in L^2(\mathbb{C}^n)$ the series

$$f = \sum_{\alpha,\beta\in\mathbb{N}^n} \langle f, \Phi_{\alpha,\beta}\rangle \Phi_{\alpha,\beta} \tag{1.1}$$

is convergent in $L^2(\mathbb{C}^n)$ and is called the *special Hermite expansion* of f (here $\langle\cdot,\cdot\rangle$ is the standard inner product in $L^2(\mathbb{C}^n)$). Denote by Q_k the spectral projection operator on the eigenspace corresponding to the k-th eigenvalue $2k + n$. Then the series (1.1) may be written in a compact form

$$f = \sum_{k=0}^{\infty} Q_k f.$$

Given functions $f, g \in L^2(\mathbb{C}^n)$, their *twisted convolution* is defined as

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - u)g(u) \exp\left(\frac{1}{2} i \operatorname{Im}\langle z, u\rangle\right) du,$$

where $\langle z, u\rangle = \sum_{j=1}^n z_j\overline{u_j}$. The above product turns $L^1(\mathbb{C}^n)$ into a (noncommutative) Banach algebra.

The spectral projections Q_k are then expressed as

$$Q_k f = f \times \phi_k^{n-1},$$

ϕ_k^{n-1} being the Laguerre functions defined by

$$\phi_k^{n-1}(z) = (2\pi)^{-n} L_k^{n-1}\left(|z|^2/2\right) \exp\left(-|z|^2/4\right), \quad z \in \mathbb{C}^n.$$

Each ϕ_k^{n-1} is an eigenfunction of $\widetilde{\Delta}$ that corresponds to the eigenvalue $2k + n$ and $\{\phi_k^{n-1} : k \in \mathbb{N}\}$ is an orthogonal (but incomplete) system in $L^2(\mathbb{C}^n)$. For all of the above and further facts regarding special Hermite expansions the reader is referred to the book of Thangavelu [**Th3**].

Let $m \geq 1$ and define

$$\vartheta_k^{m-1}(z) = \begin{cases} (2k + m)^{m-1}, & |z| \leq \sqrt{6(2k + m)}, \\ \exp(-\gamma|z|^2), & |z| > \sqrt{6(2k + m)}. \end{cases}$$

We will make use of the estimate

$$|\phi_k^{m-1}(z)| \leq c\, \vartheta_k^{m-1}, \tag{1.2}$$

which is a consequence of estimates for Laguerre functions due to Askey and Wainger [**AsWa**], compiled by Muckenhoupt [**Mu4**]. Here c and γ are independent of $k \in \mathbb{N}$ and $z \in \mathbb{C}^n$. A direct calculation using (1.2) shows that

$$\|\phi_k^{n-1}\|_{L^p} \leq c_n (2k + n)^{2n-1}, \quad 1 \leq p < \infty.$$

This estimate may be generalized by adding a proper weight to L^p norm. In fact, we have the following

PROPOSITION 1.2.1. *Let $1 \leq p < \infty$ and $\omega \in A_p(\mathbb{C}^n)$. There exists a constant c (independent of k) such that*

$$\|\phi_k^{n-1}\|_{L^p(\omega)} \leq c\,(2k+n)^{2n-1}. \tag{1.3}$$

PROOF. Denote by B_r the ball $\{z \in \mathbb{C}^n : |z| \leq r\}$. Since the A_p condition implies

$$\frac{\omega(B_r)}{\omega(B_1)} \leq c\left(\frac{|B_r|}{|B_1|}\right)^p, \quad r \geq 1,$$

we have

$$\omega(B_r) \leq cr^{2np}, \quad r \geq 1. \tag{1.4}$$

Let $\Gamma_1 = B_{\sqrt{6(2k+n)}}$ and $\Gamma_2 = \mathbb{C}^n \backslash \Gamma_1$. We decompose Γ_2 into disjoint "rings":

$$\Gamma_2 = \bigcup_{m=0}^{\infty} \Gamma_2^m, \quad \Gamma_2^m = \left\{z \in \mathbb{C}^n : 2^m\sqrt{6(2k+n)} < |z| \leq 2^{m+1}\sqrt{6(2k+n)}\right\}.$$

Now, to estimate LHS in (1.3) we split the integration over $\mathbb{C}^n$ into integration over Γ_1 and over each of Γ_2^m. By (1.2) and (1.4) we obtain

$$\begin{aligned}
\int_{\Gamma_1} |\phi_k^{n-1}(z)|^p\omega(z)\,dz &\leq c\int_{\Gamma_1}(2k+n)^{(n-1)p}\omega(z)\,dz = c(2k+n)^{(n-1)p}\omega(\Gamma_1)\\
&\leq c(2k+n)^{(n-1)p}\left(\sqrt{6(2k+n)}\right)^{2np} = c(2k+n)^{(2n-1)p},
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Gamma_2} |\phi_k^{n-1}(z)|^p\omega(z)\,dz &= \sum_{m=0}^{\infty}\int_{\Gamma_2^m}|\phi_k^{n-1}(z)|^p\omega(z)\,dz\\
&\leq c\sum_{m=0}^{\infty}\int_{\Gamma_2^m}\exp(-\gamma|z|^2p)\omega(z)\,dz\\
&\leq c\sum_{m=0}^{\infty}\exp(-\gamma p 2^{2m}6(2k+n))\int_{\Gamma_2^m}\omega(z)\,dz\\
&\leq c\sum_{m=0}^{\infty}\exp(-\gamma p 2^{2m}6(2k+n))\;\omega\left(B_{2^{m+1}\sqrt{6(2k+n)}}\right)\\
&\leq c\sum_{m=0}^{\infty}\exp(-\gamma p 2^{2m}6(2k+n))\left(2^{m+1}\sqrt{6(2k+n)}\right)^{2np}\\
&\leq c(2k+n)^{np}\sum_{m=0}^{\infty}\left(2^{2m}\right)^{np}\exp(-\gamma 2^{2m}) \leq c(2k+n)^{(2n-1)p}.
\end{aligned}$$

□

In what follows we will make use of the following lemma (cf. [**S2**], p.198).

LEMMA 1.2.2. *Assume that $\Psi\colon \mathbb{C}^n \to [0,\infty)$ is radial, and (radially) decreasing, with $\int \Psi(z)dz = 1$. Define $\Psi_\varepsilon(z) = \varepsilon^{-2n}\Psi(z/\varepsilon)$, $\varepsilon > 0$. If $1 \leq p < \infty$ and $\omega \in A_p(\mathbb{C}^n)$ then*

$$\|f * \Psi_\varepsilon\|_{L^p(\omega)} \leq A\|f\|_{L^p(\omega)}, \quad f \in L^p(\omega),$$

holds with A independent of ε and Ψ. Moreover, A depends on ω only through the A_p norm of ω.

REMARK 1.2.3. Denote by $\mathcal{R}$ the class of functions Ψ satisfying the assumptions of Lemma 1.2.2 (notice that if Ψ belongs to $\mathcal{R}$ then so does Ψ_ε). Then

$$Mf(z) = \sup_{\Psi\in\mathcal{R}} |f| * \Psi(z), \quad z \in \mathbb{C}^n,$$

where M denotes the (centered) Hardy-Littlewood maximal function in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. See [**S2**] for details.

Now we are in a position to estimate weighted L^p norms of $Q_k f$.

LEMMA 1.2.4. *Let $1 \le p < \infty$ and $\omega \in A_p$. Then*

$$\|Q_k f\|_{L^p(\omega)} \le c\,(2k+n)^{2n-1}\|f\|_{L^p(\omega)}, \quad f \in L^p(\omega),$$

with c independent of $k \in \mathbb{N}$.

PROOF. Observe that by (1.2),

$$|Q_k f(z)| = |f \times \phi_k^{n-1}(z)| \le |f| * |\phi_k^{n-1}|(z) \le c|f| * \vartheta_k^{n-1}(z),$$

and notice that ϑ_k^{n-1} is radial and radially decreasing. Thus Lemma 1.2.2 may be applied to the function $\vartheta_k^{n-1}/\|\vartheta_k^{n-1}\|_{L^1}$. As a result we obtain

$$\big\||f| * \vartheta_k^{n-1}\big\|_{L^p(\omega)} \le c\|\vartheta_k^{n-1}\|_{L^1}\|f\|_{L^p(\omega)} \le c(2k+n)^{2n-1}\|f\|_{L^p(\omega)}.$$

For the last inequality see the proof of Proposition 1.2.1. □

Our next objective is to obtain a pointwise estimate of $Q_k f$.

LEMMA 1.2.5. *Let $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. Then*

$$|Q_k f(z)| \le c\,(2k+n)^{n-1}(\sqrt{6(2k+n)} + |z|)^{2n}\|f\|_{L^p(\omega)}, \quad z \in \mathbb{C}^n,$$

with c independent of $k \in \mathbb{N}$.

PROOF. Without loss of generality we assume $f \ge 0$. For a function v on $\mathbb{C}^n$ we define

$$(\tau_\xi v)(z) = v(z-\xi), \quad \check{v}(z) = v(-z), \quad z \in \mathbb{C}^n.$$

As in the proof of Lemma 1.2.4 we have

$$|Q_k f(z)| \le cf * \vartheta_k^{n-1}(z) = c\int_{\mathbb{C}^n} f(u)\tau_z\check{\vartheta}_k^{n-1}(u)\,du.$$

To handle the integral above observe that the following estimate holds:

$$\tau_z\check{\vartheta}_k^{n-1}(u) \le \begin{cases} (2k+n)^{n-1}, & |u| \le \sqrt{6(2k+n)} + |z|, \\ \exp(-\gamma(|u|-|z|)^2), & |u| > \sqrt{6(2k+n)} + |z|. \end{cases}$$

Let $\Gamma_1 = B_{\sqrt{6(2k+n)}+|z|}$, $\Gamma_2 = \mathbb{C}^n \backslash \Gamma_1$. Further, we divide Γ_2 into disjoint "rings" Γ_2^m:

$$\Gamma_2^m = \left\{u \in \mathbb{C}^n : 2^m\sqrt{6(2k+n)} + |z| < |u| \le 2^{m+1}\sqrt{6(2k+n)} + |z|\right\}.$$

Consider the case $1 < p < \infty$. By Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{C}^n} f(u)\tau_z\check{\vartheta}_k^{n-1}(u)\,du &\le \|f\|_{L^p(\omega)}\left(\int_{\mathbb{C}^n}(\tau_z\check{\vartheta}_k^{n-1}(u))^{p'}\omega(u)^{-p'/p}\,du\right)^{1/p'} \\ &= \|f\|_{L^p(\omega)}\|\tau_z\check{\vartheta}_k^{n-1}\|_{L^{p'}(\tilde{\omega})},\end{aligned}$$

Since $\tilde{\omega} = \omega^{-p'/p}$ belongs to $A_{p'}$ (cf. [**Du**]), similarly to the proof of Proposition 1.2.1 we get

$$\begin{aligned}\int_{\Gamma_1} (\tau_z \check{\vartheta}_k^{n-1}(u))^{p'} \tilde{\omega}(u)\, du &\le (2k+n)^{(n-1)p'} \tilde{\omega}\left(B_{\sqrt{6(2k+n)}+|z|}\right) \\ &\le c(2k+n)^{(n-1)p'} \left(\sqrt{6(2k+n)}+|z|\right)^{2np'}\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_2} (\tau_z \check{\vartheta}_k^{n-1}(u))^{p'} \tilde{\omega}(u)\, du &\le \sum_{m=0}^{\infty} \int_{\Gamma_2^m} \exp(-\gamma(|u|-|z|)^2 p')\tilde{\omega}(u)\, du \\ &\le \sum_{m=0}^{\infty} \exp\left(-\gamma p' 2^{2m} 6(2k+n)\right) \tilde{\omega}\left(B_{2^{m+1}\sqrt{6(2k+n)}+|z|}\right) \\ &\le c\sum_{m=0}^{\infty} \exp\left(-\gamma 2^{2m}\right) \left(2^{m+1}\sqrt{6(2k+n)}+|z|\right)^{2np'} \\ &\le c(\sqrt{6(2k+n)}+|z|)^{2np'} \sum_{m=0}^{\infty} \exp(-\gamma 2^{2m})(2^{2m})^{2np'} \\ &= c(\sqrt{6(2k+n)}+|z|)^{2np'}.\end{aligned}$$

This proves the inequality

$$\|\tau_z \check{\vartheta}_k^{n-1}\|_{L^{p'}(\tilde{\omega})} \le c(2k+n)^{n-1}(\sqrt{6(2k+n)}+|z|)^{2n},$$

and so the assertion of the lemma is justified for $p > 1$.

If $p = 1$ then

$$\int_{\mathbb{C}^n} f(u)\tau_z \check{\vartheta}_k^{n-1}(u)\, du \le \|f\|_{L^1(\omega)} \|\tau_z \check{\vartheta}_k^{n-1}\omega^{-1}\|_\infty,$$

and it remains to estimate $\|\tau_z \check{\vartheta}_k^{n-1}\omega^{-1}\|_\infty$. In view of the A_1 condition we may write

$$\operatorname*{ess\,sup}_{u\in B_r} \frac{1}{\omega(u)} \le c\frac{|B_r|}{\omega(B_r)} \le c\frac{r^{2n}}{\omega(B_1)} \le cr^{2n}, \quad r \ge 1,$$

and therefore

$$\begin{aligned}\operatorname*{ess\,sup}_{u\in\Gamma_1} \tau_z \check{\vartheta}_k^{n-1}(u)\omega(u)^{-1} &\le (2k+n)^{n-1} \operatorname*{ess\,sup}_{u\in\Gamma_1} \omega(u)^{-1} \\ &\le c(2k+n)^{n-1}(\sqrt{6(2k+n)}+|z|)^{2n},\end{aligned}$$

$$\begin{aligned}\operatorname*{ess\,sup}_{u\in\Gamma_2^m} \tau_z \check{\vartheta}_k^{n-1}(u)\omega(u)^{-1} &\le \exp(-\gamma 2^{2m} 6(2k+n)) \operatorname*{ess\,sup}_{u\in\Gamma_2^m} \omega(u)^{-1} \\ &\le c\exp(-\gamma 2^{2m})(2^{m+1}\sqrt{6(2k+n)}+|z|)^{2n} \\ &\le c(\sqrt{6(2k+n)}+|z|)^{2n} \exp(-\gamma 2^{2m}) 2^{2nm}.\end{aligned}$$

Since

$$\|\tau_z \check{\vartheta}_k^{n-1}\omega^{-1}\|_\infty = \sup\left\{\operatorname*{ess\,sup}_{u\in\Gamma_1} \tau_z \check{\vartheta}_k^{n-1}(u)\omega(u)^{-1}, \operatorname*{ess\,sup}_{u\in\Gamma_2^m} \tau_z \check{\vartheta}_k^{n-1}(u)\omega(u)^{-1} : m \in \mathbb{N}\right\},$$

the conclusion follows. □

REMARK 1.2.6. Let f, p and ω be as in Lemma 1.2.5 and let $M \geq 0,\ m \geq 1$. Then, by the above proof and the estimate (1.2),

$$\begin{aligned}\left|f \times \left[(1+|\cdot|^2)^M \phi_k^{m-1}(\cdot)\right](z)\right| &\leq c|f| * \left[(1+|\cdot|^2)^M \vartheta_k^{m-1}(\cdot)\right](z)\\ &\leq c(2k+m)^{m-1+M}\left(\sqrt{6(2k+m)}+|z|\right)^{2n}\|f\|_{L^p(\omega)},\end{aligned}$$

with c independent of k. We will make use of this fact later.

LEMMA 1.2.7. *Let $1 \leq p < \infty$ and $\omega \in A_p$. The subspace generated by $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}^n\}$ is dense in $L^p(\omega)$ and in $C_0(\mathbb{C}^n)$ with $\|\cdot\|_\infty$ norm.*

PROOF. According to Thangavelu [**Th4**, Theorem 1.4.4], the finite linear combinations of $\Phi_{\alpha,\beta}$ are dense in the Schwartz class $\mathcal{S}(\mathbb{C}^n)$. Thus the density in C_0 follows immediately. To prove the density in $L^p(\omega)$ it is sufficient to approximate in $L^p(\omega)$ norm functions from $\mathcal{S}(\mathbb{C}^n)$ by finite linear combinations of $\Phi_{\alpha,\beta}$.

Let $f \in \mathcal{S}(\mathbb{C}^n)$. There exists $\{f_n\} \subset \operatorname{lin}\{\Phi_{\alpha,\beta}\}$ such that $f_n \to f$ in $\mathcal{S}(\mathbb{C}^n)$. It remains to show that $\|f_n - f\|_{L^p(\omega)} \to 0$. This, however, follows by Lebesgue's dominated convergence theorem, proper majorant being $c(1+|z|)^{-Np}\omega(z)$ with N sufficiently large. Indeed, we have

$$(1+|z|)^N|f_n - f| \rightrightarrows 0, \quad N \in \mathbb{N},$$

which gives $|f_n - f| \leq c(1+|z|)^{-N}$. Further, by (1.4)

$$\begin{aligned}\int_{\mathbb{C}^n}(1+|z|)^{-Np}\omega(z)\,dz &\leq \omega(B_1) + \sum_{r=2}^{\infty} r^{-Np}\omega(B_r)\\ &\leq \omega(B_1) + c\sum_{r=2}^{\infty} r^{-Np}r^{2np},\end{aligned}$$

and the last series is convergent for N sufficiently large. □

COROLLARY 1.2.8. *Let $1 \leq p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. If $\langle f, \Phi_{\alpha,\beta}\rangle = 0$ for all $\alpha, \beta \in \mathbb{N}^n$ then $f = 0$.*

PROOF. If $p = 1$ Lemma 1.2.7 gives $\langle f, g\rangle = 0$ for each $g \in C_0(\mathbb{C}^n)$ and the conclusion follows. If $1 < p < \infty$ then, again by Lemma 1.2.7, we get $\langle f, g\rangle = 0$ for each $g \in L^{p'}(\omega^{-p'/p})$ (recall that $\omega^{-p'/p} \in A_{p'}$). The claim is proved. □

Let $1 \leq p < \infty$ and $\omega \in A_p$. Given $f \in L^p(\omega)$ we define its *heat-diffusion integral* by

$$g(t,z) = \sum_{k=0}^{\infty} e^{-t(2k+n)}Q_k f(z), \quad t > 0.$$

Note, that the above series converges pointwise by Lemma 1.2.5 and in $L^p(\omega)$ by Lemma 1.2.4. We may express $g(t,z)$ as a twisted convolution with a kernel G_t by writing

$$\begin{aligned}g(t,z) &= \sum_{k=0}^{\infty} e^{-t(2k+n)} f \times \phi_k^{n-1}(z)\\ &= \sum_{k=0}^{\infty} e^{-t(2k+n)}\int_{\mathbb{C}^n} f(z-u)\phi_k^{n-1}(u)e^{i\operatorname{Im}\langle z,u\rangle/2}\,du\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{C}^n} f(z-u)\left(\sum_{k=0}^{\infty} e^{-t(2k+n)}\phi_k^{n-1}(u)\right) e^{i\,\mathrm{Im}\langle z,u\rangle/2}\,du \\
&= f \times G_t(z).
\end{aligned}
$$

Interchanging the order of summation and integration is justified by Remark 1.2.6 since

$$
\begin{aligned}
\sum_{k=0}^{\infty} e^{-t(2k+n)} \int_{\mathbb{C}^n} |f(z-u)\phi_k^{n-1}(u)|\,du &\le c\sum_{k=0}^{\infty} e^{-t(2k+n)}|f| * \vartheta_k^{n-1}(z) \\
&\le c\|f\|_{L^p(\omega)} \sum_{k=0}^{\infty} e^{-t(2k+n)}(2k+n)^{n-1}(\sqrt{6(2k+n)}+|z|)^{2n} < \infty.
\end{aligned}
$$

Using the generating formula [**Le**, (4.17.3)] we get

$$
G_t(z) = \frac{1}{(4\pi\sinh t)^n}\exp\left(-\frac{1}{4}|z|^2\coth t\right), \quad t>0.
$$

PROPOSITION 1.2.9. *Let $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. The heat-diffusion integral $g(t,z)$ of f is a smooth function on $\mathbb{R}_+ \times \mathbb{R}^{2n}$. Moreover, it satisfies*

$$
\left(\frac{\partial}{\partial t} + \widetilde{\Delta}_z\right) g(t,z) = 0. \tag{1.5}
$$

PROOF. Let E be a compact subset of $\mathbb{C}^n$. Then $\sup_{z\in E}|Q_k f(z)|$ grows polynomially in k by Lemma 1.2.5. Therefore we may differentiate term by term with respect to t the series defining $g(t,z)$. We obtain

$$
\frac{\partial^m}{\partial t^m} g(t,z) = \sum_{k=0}^{\infty}(-1)^m(2k+n)^m e^{-t(2k+n)} Q_k f(z), \tag{1.6}
$$

RHS being continuous in (t,z) since each $Q_k(z)$ is a continuous function of z and the series is convergent almost uniformly in (t,z).

For $z \in \mathbb{C}^n$ we write $z = x+iy$, $x,y\in\mathbb{R}^n$. To prove the smoothness of (1.6) in z we first focus on

$$
\partial_x^\alpha \partial_y^\beta Q_k f(z) = \partial_x^\alpha \partial_y^\beta \int_{\mathbb{C}^n} f(u)\phi_k^{n-1}(z-u)e^{i\,\mathrm{Im}\langle z,z-u\rangle/2}\,du,
$$

where $\alpha,\beta\in\mathbb{N}^n$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$ and $\partial_y^\beta = \partial_{y_1}^{\beta_1}\cdots\partial_{y_n}^{\beta_n}$. We claim that the differentiation may be taken under the integral sign. Define

$$
\begin{aligned}
\Upsilon_k^{n-1}(z,u) &= \phi_k^{n-1}(z-u)e^{i\,\mathrm{Im}\langle z,z-u\rangle/2} \\
&= (2\pi)^{-n} L_k^{n-1}\left(|z-u|^2/2\right) e^{-|z-u|^2/4 + i\,\mathrm{Im}\langle z,z-u\rangle/2}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\partial_x^\alpha \partial_y^\beta \Upsilon_k^{n-1}(z,u) \\
&\quad = e^{-|z-u|^2/4+i\,\mathrm{Im}\langle z,z-u\rangle/2} \sum_{|\nu|\le|\alpha|+|\beta|} \partial^{|\nu|} L_k^{n-1}\left(|z-u|^2/2\right)\mathcal{P}_\nu(z,z-u),
\end{aligned} \tag{1.7}
$$

where $\mathcal{P}_\nu$ are polynomials on $\mathbb{C}^n \times \mathbb{C}^n$.

Fix $z \in \mathbb{C}^n$ and set $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. For $-1 < \varepsilon < 1$, by the mean value theorem we have

$$\varepsilon^{-1} \left| \Upsilon_k^{n-1}(z + \varepsilon e_1, u) - \Upsilon_k^{n-1}(z, u) \right| = \left| \partial_x^{e_1} \Upsilon_k^{n-1}(z + \theta e_1, u) \right|$$

for some $\theta \in (-1, 1)$. In view of (1.7) the function

$$\Upsilon^*(z, \cdot) = \sup_{|\theta|<1} \left| \partial_x^{e_1} \Upsilon_k^{n-1}(z + \theta e_1, \cdot) \right|$$

is bounded and rapidly decreasing. Moreover, (1.4) implies $\Upsilon^*(z, \cdot) \in L^{p'}(\omega^{-p'/p})$ and therefore integrability of $\Upsilon^*(z,u)|f(u)|$ is justified by Hölder's inequality. Hence the dominated convergence theorem may be applied and we obtain

$$\partial_x^{e_1} Q_k f(z) = \int_{\mathbb{C}^n} f(u) \partial_x^{e_1} \Upsilon_k^{n-1}(z, u)\, du.$$

Our claim follows by repeating the above arguments for the remaining partial derivatives.

Now our aim is to show that the series

$$\sum_{k=0}^{\infty} (-1)^m (2k+n)^m e^{-t(2k+n)} \partial_x^\alpha \partial_y^\beta Q_k f(z) \tag{1.8}$$

is almost uniformly convergent. Since each term is a continuous function this will finish the proof of smoothness of $g(t, z)$ (the continuity in z of each term is checked by the mean value theorem, (1.7) and the dominated convergence theorem).

We have ([**Le**, (4.18.6)]) $\partial^m L_k^{n-1} = (-1)^m L_{k-m}^{n+m-1}$ (we use the convention that $L_{k-m}^{n+m-1} = 0$ if $k < m$). Therefore, by (1.7),

$$\left| \partial_x^\alpha \partial_y^\beta \Upsilon_k^{n-1}(z, u) \right| \;\leq\; c(1 + |z|^2)^M (1 + |z - u|^2)^M \sum_{m=0}^{|\alpha|+|\beta|} |\phi_{k-m}^{n+m-1}(z - u)|,$$

with c independent of $k \in \mathbb{N}$ and the convention that $\phi_{k-m}^{n+m-1} = 0$ for $k < m$. This, in view of Remark 1.2.6, gives

$$\left| \partial_x^\alpha \partial_y^\beta Q_k f(z) \right| \leq c \|f\|_{L^p(\omega)} (1 + |z|^2)^M (2k + n)^{n-1+|\alpha|+|\beta|+M} \left(\sqrt{6(2k + n)} + |z|\right)^{2n}.$$

Hence $\sup_{z \in E} \left| \partial_x^\alpha \partial_y^\beta Q_k f(z) \right|$ grows polynomially in k and the almost uniform convergence of (1.8) is justified.

To verify the heat equation (1.5) we differentiate term by term the series defining $g(t, z)$ and use the fact that L_k^{n-1} satisfies ([**Le**, (4.18.7)])

$$x \partial^2 L_k^{n-1}(x) + (n - x) \partial L_k^{n-1}(x) + k L_k^{n-1}(x) = 0.$$

The computation makes no difficulties and is omitted. □

THEOREM 1.2.10. *Let $1 \leq p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. Let $g(t, z)$ be the heat-diffusion integral of f. Then*

(a) $\sup_{t>0} |g(t, z)| \leq M f(z), \quad z \in \mathbb{C}^n$;
(b) $\|g(t, \cdot)\|_{L^p(\omega)} \leq C (\cosh t)^{-n} \|f\|_{L^p(\omega)}$;
(c) $\|g(t, \cdot) - f\|_{L^p(\omega)} \longrightarrow 0, \quad t \to 0^+$;
(d) $g(t, z) \longrightarrow f(z)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{T_t\}_{t>0}$, $T_t f(z) = g(t, z)$, is a strongly continuous and uniformly bounded semigroup of operators on $L^p(\omega)$.

PROOF. Observe that

$$G_t(z) = (\cosh t)^{-n} W_{\sqrt{\tanh t}}(z), \tag{1.9}$$

where $W(x) = (4\pi)^{-n}\exp(-|x|^2/4)$ is the Gauss-Weierstrass kernel in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ and $W_\varepsilon(\cdot) = \varepsilon^{-2n}W(\cdot/\varepsilon)$. Since $|g(t,z)| \le |f| * G_t(z)$ assertions (a) and (b) follow from Lemma 1.2.2 and Remark 1.2.3. Items (c) and (d) are justified by standard arguments with the aid of (a), (b) and Lemma 1.2.7.

The semigroup property of $\{T_t\}_{t>0}$ is easily verified for $f \in \operatorname{lin}\{\Phi_{\alpha,\beta}\}$; hence it holds for any $f \in L^p(\omega)$ in view of Lemma 1.2.7 and (b). Strong continuity follows by standard arguments, similarly to (c). □

We now pass to Poisson integrals. Let $1 \le p < \infty$ and $\omega \in A_p$. Given $f \in L^p(\omega)$ we define its *Poisson integral* by

$$f(t,z) = \sum_{k=0}^{\infty} e^{-t\sqrt{2k+n}} Q_k f(z), \quad t > 0.$$

The above series converges pointwise by Lemma 1.2.5 and in $L^p(\omega)$ by Lemma 1.2.4. Using the well-known formula

$$e^{-\beta t} = \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-\beta^2 s} s^{-3/2} e^{-t^2/(4s)}\, ds, \quad t > 0,\ \beta \ge 0, \tag{1.10}$$

we express $f(t,z)$ as the twisted convolution with a kernel P_t:

$$\begin{aligned} f(t,z) &= \sum_{k=0}^{\infty} e^{-t\sqrt{2k+n}} f \times \phi_k^{n-1}(z) \\ &= \sum_{k=0}^{\infty} \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-s(2k+n)} s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}\, ds \int_{\mathbb{C}^n} f(z-u)\phi_k^{n-1}(u) e^{\frac{1}{2}i\operatorname{Im}\langle z,u\rangle}\, du \\ &= \int_{\mathbb{C}^n} \frac{t}{\sqrt{4\pi}} \int_0^\infty \left(s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}} \sum_{k=0}^{\infty} e^{-s(2k+n)} \phi_k^{n-1}(u) \right) ds\, f(z-u) e^{\frac{1}{2}i\operatorname{Im}\langle z,u\rangle}\, du \qquad (1.11) \\ &= f \times P_t(z), \end{aligned}$$

where

$$P_t(z) = \frac{t}{\sqrt{4\pi}} \int_0^\infty G_s(z) s^{-3/2} e^{-t^2/(4s)}\, ds.$$

Interchanging the order of summation and integration is justified by Fubini's theorem since, by Remark 1.2.6,

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-s(2k+n)} s^{-3/2} e^{-t^2/(4s)} \int_{\mathbb{C}^n} |f(z-u)\phi_k^{n-1}(u)|\, du\, ds \\ &\qquad \le c \sum_{k=0}^{\infty} |f| * \vartheta_k^{n-1}(z) \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-s(2k+n)} s^{-3/2} e^{-t^2/(4s)}\, ds \\ &\qquad \le c\|f\|_{L^p(\omega)} \sum_{k=0}^{\infty} e^{-t\sqrt{2k+n}} (2k+n)^{n-1} \left(\sqrt{6(2k+n)} + |z| \right)^{2n} < \infty. \end{aligned}$$

Note that by (1.11) we obtain the following subordination formula:

$$f(t,z) = \frac{t}{\sqrt{4\pi}} \int_0^\infty g(s,z) s^{-3/2} e^{-t^2/(4s)}\, ds, \quad t > 0. \tag{1.12}$$

PROPOSITION 1.2.11. *Let $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. Then the Poisson integral $f(t, z)$ of f is a smooth function on $\mathbb{R}_+ \times \mathbb{R}^{2n}$. Moreover, it satisfies*

$$\left(\frac{\partial^2}{\partial t^2} - \widetilde{\Delta}_z\right) f(t, z) = 0.$$

PROOF. The proof is very similar to that of Proposition 1.2.9. We omit the details. □

THEOREM 1.2.12. *Assume that $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. Let $f(t, z)$ be the Poisson integral of f. Then*

(a) $\sup_{t>0} |f(t, z)| \le Mf(z), \quad z \in \mathbb{C}^n$;

(b) $\|f(t, \cdot)\|_{L^p(\omega)} \le Ce^{-t\sqrt{n}}\|f\|_{L^p(\omega)}$;

(c) $\|f(t, \cdot) - f\|_{L^p(\omega)} \longrightarrow 0, \quad t \to 0^+$;

(d) $f(t, z) \longrightarrow f(z)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{P_t\}_{t>0}$, $P_t f(z) = f(t, z)$, is a strongly continuous and uniformly bounded semigroup of operators on $L^p(\omega)$.

PROOF. Using the subordination formula (1.12) and (a) of Theorem 1.2.10 we get

$$|f(t, z)| \le \int_0^\infty Mf(z) \frac{t}{\sqrt{4\pi}} s^{-3/2} e^{-t^2/(4s)}\, ds$$

and so (a) follows. To prove (b) we apply Minkowski's integral inequality and Theorem 1.2.10 (b) to obtain

$$\begin{aligned} \|f(t, \cdot)\|_{L^p(\omega)} &\le \int_0^\infty \|g(s, \cdot)\|_{L^p(\omega)} \frac{t}{\sqrt{4\pi}} s^{-3/2} e^{-t^2/(4s)}\, ds \\ &\le c\|f\|_{L^p(\omega)} \int_0^\infty (\cosh s)^{-n} \frac{t}{\sqrt{4\pi}} s^{-3/2} e^{-t^2/(4s)}\, ds \\ &\le ce^{-t\sqrt{n}}\|f\|_{L^p(\omega)}. \end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 1.2.10. □

REMARK 1.2.13. Theorem 1.2.10 shows that the condition $\omega \in A_p$ is sufficient to have $\|g(t, z)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$. However, it is not necessary. For $1 < p < \infty$ it was proved in [**KeTh**] that a certain local A_p condition is both necessary and sufficient for the above weighted norm inequality to hold. Some results of this type were also obtained in the case of the one-dimensional system of Laguerre functions defined in Remark 1.5.8 below.

REMARK 1.2.14. Most of the results of this section hold for the space $L^\infty(\mathbb{C}^n)$. More precisely, Proposition 1.2.1, Lemma 1.2.4 (and so Lemma 1.2.5), Proposition 1.2.9 and Proposition 1.2.11 remain valid if we replace $L^p(\omega)$ by L^∞. Moreover, Theorems 1.2.10 and 1.2.12, except (c) and (d), also remain valid with L^∞ replacing $L^p(\omega)$. Concerning (c) and (d), we have $\|g(t, \cdot) - f\|_\infty \to 0$ and $\|f(t, \cdot) - f\|_\infty \to 0$, $t \to 0^+$, but only for $f \in C_0(\mathbb{C}^n)$.

REMARK 1.2.15. If $1 \le p \le \infty$ and $f \in L^p(\mathbb{C}^n)$ (the case $\omega \equiv 1$) then Theorem 1.2.10 (b) is valid with $C = 1$ and Theorem 1.2.12 (b) holds with $Ce^{-t\sqrt{n}}$ dropped. In particular, this means that $\{T_t\}$ and $\{P_t\}$ are semigroups of contractions on $L^p(\mathbb{C}^n)$, $1 \le p \le \infty$.

1.3. Laguerre function expansions; system $\{\ell_k^\alpha\}$

Let $k=(k_1,\dots,k_d)\in\mathbb{N}^d$ and $\alpha=(\alpha_1,\dots,\alpha_d)\in(-1,\infty)^d$ be multi-indices. The Laguerre function ℓ_k^α on $\mathbb{R}^d_+$ is defined as

$$\ell_k^\alpha(x)=\ell_{k_1}^{\alpha_1}(x_1)\cdot\ldots\cdot\ell_{k_d}^{\alpha_d}(x_d),\qquad x=(x_1,\dots,x_d)\in\mathbb{R}^d_+,$$

where $\ell_{k_i}^{\alpha_i}$ are one-dimensional Laguerre functions given by

$$\ell_{k_i}^{\alpha_i}(x_i)=\left(\frac{\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)}\right)^{1/2}L_{k_i}^{\alpha_i}(x_i)e^{-x_i/2},\qquad x_i>0,\quad i=1,\dots,d.$$

Each ℓ_k^α is an eigenfunction of the differential operator

$$L=\sum_{i=1}^d\left(x_i\frac{\partial^2}{\partial x_i^2}+(\alpha_i+1)\frac{\partial}{\partial x_i}-\frac{x_i}{4}\right),$$

the corresponding eigenvalue being $-|k|-(|\alpha|+d)/2$. The operator $-L$ is positive and symmetric in $L^2(\mathbb{R}^d_+,x^\alpha dx)$. Moreover, the system $\{\ell_k^\alpha:k\in\mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d_+,x^\alpha dx)$.

The following estimate of ℓ_k^α is crucial for further considerations:

$$|\ell_k^\alpha(x)|\le c\prod_{i=1}^d\Theta_{k_i}^{\alpha_i}(x_i),\qquad x\in\mathbb{R}^d_+,\tag{1.13}$$

where

$$\Theta_{k_i}^{\alpha_i}(x_i)=\begin{cases}(2k_i+|\alpha_i|+1)^{|\alpha_i|/2}, & 0<x_i\le 3(2k_i+|\alpha_i|+1),\\ \exp(-\gamma x_i), & x_i>3(2k_i+|\alpha_i|+1).\end{cases}$$

Here c and γ are independent of k and x. The estimate (1.13) is a consequence of Muckenhoupt's generalization [**Mu4**] of the classical estimates due to Askey and Wainger [**AsWa**].

Let $1\le p<\infty$. We denote by $A_p^\alpha=A_p(\mathbb{R}^d_+,d\eta_\alpha)$ the class of A_p weights on $\mathbb{R}^d_+$ with respect to the (doubling) measure $\eta_\alpha(dx)=x^\alpha dx$. More precisely, A_p^α is the class of all nonnegative functions $\omega\in L^1_{\text{loc}}(\mathbb{R}^d_+,d\eta_\alpha)$, such that $\omega^{-p'/p}\in L^1_{\text{loc}}(\mathbb{R}^d_+,d\eta_\alpha)$ and

$$\sup_{Q\in\mathcal{B}}\left[\frac{1}{\eta_\alpha(Q)}\int_Q\omega(x)\eta_\alpha(dx)\right]\left[\frac{1}{\eta_\alpha(Q)}\int_Q\omega(x)^{-p'/p}\eta_\alpha(dx)\right]^{p/p'}<\infty\tag{1.14}$$

if $1<p<\infty$, or

$$\sup_{Q\in\mathcal{B}}\frac{1}{\eta_\alpha(Q)}\int_Q\omega(x)\eta_\alpha(dx)\ \operatorname*{ess\,sup}_{x\in Q}\frac{1}{\omega(x)}<\infty\tag{1.15}$$

if $p=1$. Here $\mathcal{B}$ denotes the class of all sets of the form $Q=\widetilde{Q}\cap\mathbb{R}^d_+$, where $\widetilde{Q}$ is a cube (with sides parallel to the coordinate axes) in $\mathbb{R}^d$ with center in $\mathbb{R}^d_+$.

For $r>0$ denote by Q_r the cube $(0,r)^d$. Given $1\le p<\infty$ and $\omega\in A_p^\alpha$ we have

$$\omega(Q_r)=\int_{Q_r}\omega(x)\eta_\alpha(dx)\le cr^{(d+|\alpha|)p},\qquad r\ge 1.\tag{1.16}$$

Indeed, if $1<p<\infty$ and $r\ge 1$ then, by Hölder's inequality and the A_p^α condition,

$$\eta_\alpha(Q_1)\ =\ \int_{Q_1}\omega(x)^{1/p}\omega(x)^{-1/p}\eta_\alpha(dx)$$

$$\begin{aligned} &\le \Big(\int_{Q_1}\omega(x)\eta_\alpha(dx)\Big)^{1/p}\Big(\int_{Q_1}\omega(x)^{-p'/p}\eta_\alpha(dx)\Big)^{1/p'} \\ &\le c\omega(Q_1)^{1/p}\eta_\alpha(Q_r)\omega(Q_r)^{-1/p}. \end{aligned}$$

If $p=1$ then the A_1^α condition (1.15) gives

$$\eta_\alpha(Q_1)\le\int_{Q_1}\omega(x)\eta_\alpha(dx)\operatorname*{ess\,sup}_{x\in Q_r}\frac{1}{\omega(x)}\le c\omega(Q_1)\eta_\alpha(Q_r)\omega(Q_r)^{-1}.$$

Since $\eta_\alpha(Q_r)=cr^{d+|\alpha|}$ the inequality in (1.16) follows.

LEMMA 1.3.1. *Let $1\le p<\infty$ and $\omega\in A_p^\alpha$. There exist constants $\delta=\delta_\alpha$ and c independent of $k\in\mathbb{N}^d$ such that*

$$\|\ell_k^\alpha\|_{L^p(\omega d\eta_\alpha)}\le c(2|k|+\|\alpha\|+1)^{\delta d},$$

where $\|\alpha\|=\sum_{i=1}^d|\alpha_i|$.

PROOF. Set $\lambda_i=3(2k_i+|\alpha_i|+1)$, $i=1,\dots,d$ and $\lambda^*=\max\{\lambda_i:1\le i\le d\}$. For $S\subset\{1,\dots,d\}$ we define

$$\Gamma_k^\alpha(S)=\{x\in\mathbb{R}_+^d: x_j>\lambda_j \text{ for } j\in S \ \&\ x_j\le\lambda_j \text{ for } j\notin S\}.$$

Note that $\{\Gamma_k^\alpha(S):S\subset\{1,\dots,d\}\}$ is a decomposition of $\mathbb{R}_+^d$ into 2^d disjoint subsets. Therefore, to finish the proof it is sufficient to obtain a proper estimate of

$$I_k^\alpha(S)=\int_{\Gamma_k^\alpha(S)}|\ell_k^\alpha(x)|^p\omega(x)\eta_\alpha(dx).$$

If $S=\emptyset$ then by (1.13) and (1.16) we have

$$\begin{aligned} I_k^\alpha(S) &\le c\int_{\Gamma_k^\alpha(S)}\Big(\max_{1\le i\le d}\lambda_i^{|\alpha_i|/2}\Big)^{dp}\omega(x)\eta_\alpha(dx) \\ &\le c(2|k|+\|\alpha\|+1)^{\|\alpha\|dp/2}\int_{Q_{\lambda^*}}\omega(x)\eta_\alpha(dx) \\ &\le c(2|k|+\|\alpha\|+1)^{\|\alpha\|dp/2}(\lambda^*)^{(d+|\alpha|)p}\le c(2|k|+\|\alpha\|+1)^{(1+3\|\alpha\|/2)dp}. \end{aligned}$$

If $S\ne\emptyset$ we divide $\Gamma_k^\alpha(S)$ into disjoint subsets $\Gamma_k^\alpha(S,m)$, $m\in\mathbb{N}$, defined as follows:

$$\Gamma_k^\alpha(S,0)=\widetilde{\Gamma}_k^\alpha(S,0),\quad \Gamma_k^\alpha(S,m)=\widetilde{\Gamma}_k^\alpha(S,m)\backslash\widetilde{\Gamma}_k^\alpha(S,m-1),\quad m\ge1,$$

where

$$\widetilde{\Gamma}_k^\alpha(S,m)=\{x\in\mathbb{R}_+^d:\lambda_j<x_j\le2^{m+1}\lambda_j \text{ for } j\in S \ \&\ x_j\le\lambda_j \text{ for } j\notin S\}.$$

Now, using (1.13) and (1.16) we obtain

$$\begin{aligned} I_k^\alpha(S) &\le c\sum_{m=0}^\infty\int_{\Gamma_k^\alpha(S,m)}\Big[\exp\Big(-\gamma\sum_{j\in S}x_j\Big)\prod_{j\notin S}(\lambda_j)^{|\alpha_j|/2}\Big]^p\omega(x)\eta_\alpha(dx) \\ &\le c\prod_{j=1}^d(\lambda_j)^{|\alpha_j|p/2}\sum_{m=0}^\infty\exp\Big(-\gamma p2^m\min_{j\in S}\lambda_j\Big)\int_{\Gamma_k^\alpha(S,m)}\omega(x)\eta_\alpha(dx) \\ &\le c(2|k|+\|\alpha\|+1)^{\|\alpha\|dp/2}\sum_{m=0}^\infty\exp(-\gamma p2^m)\int_{Q_{2^{m+1}\lambda^*}}\omega(x)\eta_\alpha(dx) \end{aligned}$$

$$\begin{aligned}
&\le c(2|k|+||\alpha||+1)^{||\alpha||dp/2}\sum_{m=0}^{\infty}\exp\left(-\gamma p2^m\right)\left(2^{m+1}\lambda^*\right)^{(d+|\alpha|)p}\\
&\le c(2|k|+||\alpha||+1)^{||\alpha||dp/2+(d+|\alpha|)p}\sum_{m=0}^{\infty}2^{m(d+|\alpha|)p}\exp\left(-\gamma p2^m\right)\\
&\le c(2|k|+||\alpha||+1)^{(1+3||\alpha||/2)dp}.
\end{aligned}$$

The conclusion follows. □

LEMMA 1.3.2. *Let $1 \le p < \infty$ and $\omega \in A_p^\alpha$. The Fourier-Laguerre coefficients $\langle \ell_k^\alpha, f\rangle = \int_{\mathbb{R}_+^d} \ell_k^\alpha(x) f(x)\eta_\alpha(dx)$ exist for $f \in L^p(\omega d\eta_\alpha)$. Moreover, there exist constants $\delta = \delta_\alpha$ and c independent of $k \in \mathbb{N}^d$ such that*

$$|\langle \ell_k^\alpha, f\rangle| \le c(2|k|+||\alpha||+1)^{\delta d}\|f\|_{L^p(\omega d\eta_\alpha)}. \tag{1.17}$$

PROOF. For $1<p<\infty$ Hölder's inequality implies

$$|\langle \ell_k^\alpha, f\rangle| \le \left[\int_{\mathbb{R}_+^d} |\ell_k^\alpha(x)|^{p'}\omega(x)^{-p'/p}\eta_\alpha(dx)\right]^{1/p'}\|f\|_{L^p(\omega d\eta_\alpha)},$$

and since $\omega^{-p'/p} \in A_{p'}^\alpha$ Lemma 1.3.1 gives (1.17).

The case $p=1$ is less straightforward. We use the notation from the proof of Lemma 1.3.1. We have

$$\begin{aligned}
|\langle \ell_k^\alpha, f\rangle| &\le \sum_{S\subset\{1,\dots,d\}} \operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S)} \frac{1}{\omega(y)}|\ell_k^\alpha(y)| \int_{\Gamma_k^\alpha(S)} |f(x)|\omega(x)\eta_\alpha(dx)\\
&\le \|f\|_{L^1(\omega d\eta_\alpha)} \max_{S\subset\{1,\dots,d\}} \operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S)} \frac{1}{\omega(y)}|\ell_k^\alpha(y)|.
\end{aligned}$$

If $S=\emptyset$ then by the A_1^α condition, (1.13) and (1.16) we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S)} \frac{1}{\omega(y)}|\ell_k^\alpha(y)| &\le c(2|k|+||\alpha||+1)^{||\alpha||d/2}\operatorname*{ess\,sup}_{y\in Q_{\lambda^*}}\frac{1}{\omega(y)}\\
&\le c(2|k|+||\alpha||+1)^{||\alpha||d/2}\frac{\eta_\alpha(Q_{\lambda^*})}{\omega(Q_{\lambda^*})}\\
&\le c(2|k|+||\alpha||+1)^{||\alpha||d/2}\frac{(\lambda^*)^{d+|\alpha|}}{\omega(Q_1)}\\
&\le c(2|k|+||\alpha||+1)^{(1+3||\alpha||/2)d}.
\end{aligned}$$

If $S\neq\emptyset$ we use again the A_1^α condition, (1.13) and (1.16) to get

$$\begin{aligned}
\operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S)} \frac{1}{\omega(y)}|\ell_k^\alpha(y)| &\le \sup_{m\in\mathbb{N}}\operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S,m)}\frac{1}{\omega(y)}|\ell_k^\alpha(y)|\\
&\le c\sup_{m\in\mathbb{N}}\operatorname*{ess\,sup}_{y\in\Gamma_k^\alpha(S,m)}\frac{1}{\omega(y)}\exp\Big(-\gamma\sum_{j\in S}y_j\Big)\prod_{j\notin S}\lambda_j^{|\alpha_j|/2}\\
&\le c\prod_{j=1}^{d}\lambda_j^{|\alpha|/2}\sup_{m\in\mathbb{N}}\exp\Big(-\gamma 2^m\min_{j\in S}\lambda_j\Big)\operatorname*{ess\,sup}_{y\in Q_{2^{m+1}\lambda^*}}\frac{1}{\omega(y)}\\
&\le c(2k+||\alpha||+1)^{||\alpha||d/2}\sup_{m\in\mathbb{N}}\exp(-\gamma 2^m)\frac{\eta_\alpha(Q_{2^{m+1}\lambda^*})}{\omega(Q_{2^{m+1}\lambda^*})}
\end{aligned}$$

$$\begin{aligned}&\le c(2k+\|\alpha\|+1)^{\|\alpha\|d/2}\sup_{m\in\mathbb{N}}\frac{(2^{m+1}\lambda^*)^{d+|\alpha|}}{\omega(Q_1)}\exp(-\gamma 2^m)\\ &\le c(2k+\|\alpha\|+1)^{\|\alpha\|d/2+d+|\alpha|}\sup_{m\in\mathbb{N}}(2^m)^{d+|\alpha|}\exp(-\gamma 2^m)\\ &\le c(2k+\|\alpha\|+1)^{(1+3\|\alpha\|/2)d}.\end{aligned}$$

The proof is finished. □

LEMMA 1.3.3. *Let $1\le p<\infty$ and $\omega\in A_p^\alpha$. The subspace spanned by $\{\ell_k^\alpha : k\in\mathbb{N}^d\}$ is dense in $L^p(\omega d\eta_\alpha)$ and in $C_0(\mathbb{R}_+^d)$ with $\|\cdot\|_\infty$ norm.*

PROOF. It is sufficient to approximate functions from $C_c^\infty(\mathbb{R}_+^d)$ by linear combinations of ℓ_k^α. We consider first the case of $L^p(\omega d\eta_\alpha)$.

Fix $f\in C_c^\infty(\mathbb{R}_+^d)$ and define

$$S_N f=\sum_{|k|\le N}\langle\ell_k^\alpha,f\rangle\ell_k^\alpha.$$

We will show that there exists a subsequence of $\{S_N f\}$ convergent to f in $L^p(\omega d\eta_\alpha)$. Since $S_N f\longrightarrow f$ in $L^2(d\eta_\alpha)$, there exists a subsequence $S_{N_k}f$ convergent to f $\eta_\alpha-$ *a.e.* and thus $S_{N_k}f\longrightarrow f$ *a.e.* Next, observe that by the symmetry of $-L$ we have, for $m\in\mathbb{N}$,

$$(1.18)\qquad \langle\ell_k^\alpha,f\rangle=\langle(-L)^{-m}\ell_k^\alpha,(-L)^m f\rangle=\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}\langle\ell_k^\alpha,(-L)^m f\rangle,$$

hence, by the Schwarz inequality,

$$(1.19)\qquad |S_N f(x)|\le\sum_{k\in\mathbb{N}^d}\|(-L)^m f\|_{L^2(d\eta_\alpha)}\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}|\ell_k^\alpha(x)|.$$

Therefore, for $1<p<\infty$, by Hölder's inequality we get

(1.20)

$$|S_N f(x)|^p\le c\Big[\sum_{k\in\mathbb{N}^d}\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}\Big]^{p/p'}\sum_{k\in\mathbb{N}^d}\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}|\ell_k^\alpha(x)|^p.$$

Now, Lemma 1.3.1 implies, for m sufficiently large,

$$\begin{aligned}\int_{\mathbb{R}_+^d}|S_N f(x)|^p\omega(x)\eta_\alpha(dx)&\le c\sum_{k\in\mathbb{N}^d}\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}\|\ell_k^\alpha\|^p_{L^p(\omega d\eta_\alpha)}\\ &\le c\sum_{k\in\mathbb{N}^d}\left(|k|+\frac{|\alpha|+d}{2}\right)^{-m}(2|k|+\|\alpha\|+1)^{\delta dp}<\infty.\end{aligned}$$

To show that $\|S_{N_k}f-f\|_{L^p(\omega d\eta_\alpha)}\longrightarrow 0$ as $k\to\infty$ we apply the dominated convergence theorem (the majorant is $(\Upsilon+|f|^p)\omega$, where Υ is the RHS in (1.19) if $p=1$ or in (1.20) if $p>1$).

We pass to the case of $C_0(\mathbb{R}_+^d)$. Since $S_{N_k}f\longrightarrow f$ *a.e.*, the proof is finished once we show that $S_N f$ is uniformly fundamental.

Let $1\le N<M$. By (1.13), (1.18) and the Schwarz inequality we obtain

$$|S_M f(x)-S_N f(x)|\le\sum_{n=N+1}^{M}\sum_{|k|=n}|\langle\ell_k^\alpha,f\rangle||\ell_k^\alpha(x)|$$

$$\le c \sum_{n=N+1}^{M} \sum_{|k|=n} \|(-L)^m f\|_{L^2(d\eta_\alpha)} \left(|k| + \frac{|\alpha|+d}{2}\right)^{-m} (2|k| + ||\alpha|| + 1)^{||\alpha||d/2}$$

$$\le c \sum_{n=N+1}^{M} \left(n + \frac{|\alpha|+d}{2}\right)^{-m} \left(n + \frac{||\alpha||+1}{2}\right)^{||\alpha||d/2} n^d.$$

The last expression tends to 0 as $N, M \to \infty$, if only m is chosen sufficiently large. □

COROLLARY 1.3.4. *Let* $1 \le p < \infty$, $\omega \in A_p^\alpha$ *and* $f \in L^p(\omega d\eta_\alpha)$. *If* $\langle \ell_k^\alpha, f\rangle = 0$ *for all* $k \in \mathbb{N}^d$ *then* $f = 0$.

PROOF. Apply the arguments from the proof of Corollary 1.2.8. □

Let $1 \le p < \infty$ and $\omega \in A_p^\alpha$. Given $f \in L^p(\omega d\eta_\alpha)$ we define its *heat-diffusion integral* by

$$g^\alpha(t,x) = \sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} \langle \ell_k^\alpha, f\rangle \ell_k^\alpha(x), \quad t>0.$$

The above series converges, since by (1.13) and Lemma 1.3.2

$$(1.21)\quad \sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} |\langle \ell_k^\alpha, f\rangle||\ell_k^\alpha(x)|$$

$$\le c\|f\|_{L^p(\omega d\eta_\alpha)} \sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} (2|k| + ||\alpha|| + 1)^{\delta d} \prod_{i=1}^{d} (2k_i + |\alpha_i| + 1)^{|\alpha_i|/2}$$

$$\le c \sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} (2n + ||\alpha|| + 1)^{\delta d + ||\alpha||d/2} n^d < \infty.$$

To obtain an integral form of $g^\alpha(t,x)$ we write

$$\begin{aligned} g^\alpha(t,x) &= \sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} \ell_k^\alpha(x) \int_{\mathbb{R}_+^d} \ell_k^\alpha(y) f(y) \eta_\alpha(dy) \\ &= \int_{\mathbb{R}_+^d} \left(\sum_{n=0}^{\infty} e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} \ell_k^\alpha(x)\ell_k^\alpha(y) \right) f(y)\eta_\alpha(dy) \\ &= \int_{\mathbb{R}_+^d} G_t^\alpha(x,y) f(y) \eta_\alpha(dy). \end{aligned}$$

Interchanging the order of summation and integration is easily justified by (1.13) and Lemma 1.3.2 (see (1.21)).

The kernel $G_t^\alpha(x,y)$ may be computed explicitly since a proper generating formula is available (cf. [**Le**, (4.17.6)]). The result is

$$G_t^\alpha(x,y) = \left(2\sinh\frac{t}{2}\right)^{-d} \exp\left(-\frac{1}{2}\coth\frac{t}{2}\sum_{i=1}^{d}(x_i+y_i)\right) \prod_{i=1}^{d} (\sqrt{x_i y_i})^{-\alpha_i} I_{\alpha_i}\left(\frac{\sqrt{x_i y_i}}{\sinh\frac{t}{2}}\right),$$

where $x, y \in \mathbb{R}_+^d$ and $I_a(s) = i^{-a} J_a(is)$ is the Bessel function of an imaginary argument, cf. [**Le**]. In particular, it follows that $G_t^\alpha(x,y)$ is strictly positive for $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$.

PROPOSITION 1.3.5. *Let* $1 \le p < \infty$, $\omega \in A_p^\alpha$ *and* $f \in L^p(\omega d\eta_\alpha)$. *The heat-diffusion integral* $g^\alpha(t,x)$ *of* f *is a* C^∞ *function on* $\mathbb{R}_+ \times \mathbb{R}_+^d$. *Moreover, it satisfies*

$$\left(L_x - \frac{\partial}{\partial t}\right) g^\alpha(t,x) = 0. \tag{1.22}$$

PROOF. Since $\sum_{|k|=n} |\langle \ell_k^\alpha, f\rangle \ell_k^\alpha(x)|$ grows polynomially in n, uniformly with respect to x, (see (1.21)), we may differentiate in t term by term the series defining $g^\alpha(t,x)$. The result is

$$\frac{\partial^m}{\partial t^m} g^\alpha(t,x) = \sum_{n=0}^\infty (-1)^m \left(n + \frac{|\alpha|+d}{2}\right)^m e^{-t(n+(|\alpha|+d)/2)} \sum_{|k|=n} \langle \ell_k^\alpha, f\rangle \ell_k^\alpha(x), \tag{1.23}$$

RHS being continuous since the series converges almost uniformly in (t,x).

Using the formula (cf. [**Le**, (4.18.6)])

$$\frac{\partial}{\partial x_j} \ell_k^\alpha(x) = -\sqrt{k_j}\, \ell_{k-e_j}^{\alpha+e_j}(x) - \frac{1}{2} \ell_k^\alpha(x), \quad k_j > 0,$$

together with (1.13) we obtain

$$\left| \frac{\partial}{\partial x_j} \ell_k^\alpha(x) \right| \le c\,(2|k| + ||\alpha|| + 1)^{||\alpha||/2+1}.$$

Thus $\sum_{|k|=n} |\langle \ell_k^\alpha, f\rangle \partial_{x_j} \ell_k^\alpha(x)|$ grows polynomially in n, uniformly with respect to x. Therefore we may differentiate in x_j term by term the series in (1.23), the result being a continuous function since the convergence is almost uniform in (t,x). The same arguments apply to higher derivatives, so $g^\alpha(t,x)$ is smooth on $\mathbb{R}_+ \times \mathbb{R}_+^d$.

The heat equation (1.22) is easily verified by differentiating term by term the series of $g^\alpha(t,x)$. □

Denote by M_s^α the strong maximal function in $\mathbb{R}_+^d$ with respect to the measure η_α, i.e. given $f \in L^1_{\text{loc}}(\mathbb{R}_+^d, d\eta_\alpha)$ we have

$$M_s^\alpha f(x) = \sup_{x \in H \in \mathcal{H}} \frac{1}{\eta_\alpha(H)} \int_H |f(y)| \eta_\alpha(dy),$$

where $\mathcal{H}$ is the family of all "rectangles" in $\mathbb{R}_+^d$ with sides parallel to the coordinate axes.

For $1 \le p < \infty$ we denote by $(A_p^\alpha)^* = A_p^*(\mathbb{R}_+^d, d\eta_\alpha)$ the strong A_p class of weights in $\mathbb{R}_+^d$ with respect to the measure η_α. More precisely, $(A_p^\alpha)^*$ consists of those functions from A_p^α which satisfy the condition (1.14) if $p > 1$, or (1.15) if p=1, with the supremum taken over $\mathcal{H}$. We note that if $p > 1$ then $\omega \in (A_p^\alpha)^*$ if and only if M_s^α is bounded on $L^p(\mathbb{R}_+^d, \omega d\eta_\alpha)$. This seems to be well-known and follows by an adaptation of the proof for the Lebesgue measure case, which may be found for instance in [**GaRu**].

THEOREM 1.3.6. *Let* $1 < p < \infty$, $\omega \in (A_p^\alpha)^*$ *and* $f \in L^p(\omega d\eta_\alpha)$. *Let* $g^\alpha(t,x)$ *be the heat-diffusion integral of* f. *Then*

(a) $\sup_{t>0} |g^\alpha(t,x)| \le C M_s^\alpha f(x), \quad x \in \mathbb{R}_+^d$;
(b) $\|g^\alpha(t,\cdot)\|_{L^p(\omega d\eta_\alpha)} \le C \exp(-t(|\alpha|+d)/2) \|f\|_{L^p(\omega d\eta_\alpha)}$;
(c) $\|g^\alpha(t,\cdot) - f\|_{L^p(\omega d\eta_\alpha)} \longrightarrow 0, \quad t \to 0^+$;
(d) $g^\alpha(t,x) \longrightarrow f(x)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{T_t^\alpha\}_{t>0}$, $T_t^\alpha f(x) = g^\alpha(t,x)$, *is a strongly continuous and uniformly bounded semigroup of operators on* $L^p(\omega d\eta_\alpha)$.

To prove the theorem we will need a multi-dimensional analogue of a result used by Muckenhoupt. Although the proof is a straightforward modification of that in [**Mu1**], we give it for the sake of completeness.

LEMMA 1.3.7. *Let* η *be a positive, absolutely continuous measure on* $\mathbb{R}_+^d$. *Assume that* f *is a measurable function on* $\mathbb{R}_+^d$, $g \in L^1(\mathbb{R}_+^d, d\eta)$, $g \geq 0$ *and* $g(y) = g_1(y_1)\cdots g_d(y_d)$, $y \in \mathbb{R}_+^d$. *Suppose also that for some* $x \in \mathbb{R}_+^d$ *each* $g_i(\cdot)$ *is increasing for* $y_i < x_i$ *and decreasing for* $y_i > x_i$. *Then*

$$\int_{\mathbb{R}_+^d} |f(y)|g(y)\,\eta(dy) \leq \|g\|_{L^1(d\eta)} M_s^\eta f(x),$$

where M_s^η *denotes the strong maximal function associated with the measure* η.

PROOF. Given $y, x \in \mathbb{R}_+^d$ denote by $[y,x]$ the d-dimensional interval

$$[\min(x_1,y_1), \max(x_1,y_1)] \times \cdots \times [\min(x_d,y_d), \max(x_d,y_d)].$$

Observe that each g_i may be approximated by an increasing sequence of simple functions of the form

$$\sum_j a_j \mathbf{1}_{[u^j,x_i]}(y_i) + \sum_j b_j \mathbf{1}_{[x_i,v^j]}(y_i), \quad a_j, b_j > 0,$$

where $\mathbf{1}_{(\cdot)}$ stands for the indicator function. Thus g may be approximated by an increasing sequence of functions of the form

$$\tilde{g}(y) = \sum_j c_j \mathbf{1}_{[z^j,x]}(y), \quad c_j > 0.$$

But for such $\tilde{g}$ we have

$$\begin{aligned}
\int_{\mathbb{R}_+^d} |f(y)|\tilde{g}(y)\eta(dy) &= \sum_j c_j \int_{[z^j,x]} |f(y)|\eta(dy) \\
&= \sum_j c_j \eta([z^j,x]) \frac{1}{\eta([z^j,x])} \int_{[z^j,x]} |f(y)|\eta(dy) \\
&\leq M_s^\eta f(x) \sum_j c_j \eta([z^j,x]) = M_s^\eta f(x) \|g\|_{L^1(d\eta)}.
\end{aligned}$$

□

PROOF OF THEOREM 1.3.6. Observe that $G_t^\alpha(x,y) = \prod_{i=1}^d G_t^{\alpha_i}(x_i,y_i)$, where $G_t^{\alpha_i}(x_i,y_i) \geq 0$. According to [**St**, Lemma 2.2] there exists a function $K(t,x,y) = \prod_{i=1}^d K_i(t,x_i,y_i)$ with the following properties:

(i) $G_t^{\alpha_i}(x_i,y_i) \leq \exp(-t(\alpha_i+1)/2)\, K_i(t,x_i,y_i), \quad i = 1,\ldots,d;$

(ii) for each $t > 0$ and $x_i > 0$, $K_i(t,x_i,y_i)$ as a function of y_i is increasing on $[0,x_i]$ and decreasing on $[x_i,\infty)$;

(iii) $\int_0^\infty K_i(t,x_i,y_i) y_i^{\alpha_i} dy_i \leq C$ independently of x_i and $t > 0$.

Thus by Lemma 1.3.7,

$$|g^\alpha(t,x)| \leq Ce^{-t(|\alpha|+d)/2} M_s^\alpha f(x), \quad x \in \mathbb{R}_+^d, \tag{1.24}$$

and hence (a) and (b) follow. (c) and (d) are justified by standard arguments with the aid of (a), (b) and Lemma 1.3.3.

The semigroup property is easily verified to hold for any ℓ_k^α; hence by (b) and Lemma 1.3.3 it holds for all $f \in L^p(\omega d\eta_\alpha)$. Strong continuity follows by standard reasoning, similarly to (c). □

PROPOSITION 1.3.8. *Let* $1 \le p, q < \infty$, $\omega \in A_p^\alpha$, $\rho \in A_q^\alpha$ *and* $f \in L^p(\omega d\eta_\alpha)$. *Then*

$$\|T_t^\alpha f\|_{L^q(\rho d\eta_\alpha)} \le C(t)\|f\|_{L^p(\omega d\eta_\alpha)},$$

where $C(t)$, $t > 0$, *is a continuous and decreasing function of* t *that vanishes at infinity.*

PROOF. Using Lemmas 1.3.1 and 1.3.2 we write

$$\begin{aligned}\|T_t^\alpha f\|_{L^q(\rho d\eta_\alpha)} &\le \sum_{n=0}^{\infty} e^{-t(n+(d+|\alpha|)/2)} \sum_{|k|=n} |\langle \ell_k^\alpha, f\rangle| \|\ell_k^\alpha\|_{L^q(\rho d\eta_\alpha)} \\ &\le c\left(\sum_{n=0}^{\infty} e^{-t(n+(d+|\alpha|)/2)}(2n+\|\alpha\|+1)^{2\delta d} n^d\right)\|f\|_{L^p(\omega d\eta_\alpha)}.\end{aligned}$$

□

COROLLARY 1.3.9. *Let* $1 \le p < \infty$ *and* $\omega \in A_p^\alpha$. *The family* $\{T_t^\alpha\}_{t>0}$ *is a strongly continuous semigroup of operators on* $L^p(\omega d\eta_\alpha)$ *(note that the continuity at* 0^+ *is not postulated here).*

We now pass to Poisson integrals. Let $1 \le p < \infty$ and $\omega \in A_p^\alpha$. Given $f \in L^p(\omega d\eta_\alpha)$ we define its *Poisson integral* by

$$f^\alpha(t,x) = \sum_{n=0}^{\infty} e^{-t\sqrt{n+(|\alpha|+d)/2}} \sum_{|k|=n} \langle \ell_k^\alpha, f\rangle \ell_k^\alpha(x), \quad t > 0.$$

The above series converges (see (1.21)). Using (1.10) we obtain an integral form of $f^\alpha(t,x)$:

$$\begin{aligned} f^\alpha(t,x) &= \sum_{n=0}^{\infty} \frac{t}{\sqrt{4\pi}} \int_0^\infty e^{-s\left(n+\frac{|\alpha|+d}{2}\right)} s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}\, ds \sum_{|k|=n} \ell_k^\alpha(x) \int_{\mathbb{R}_+^d} \ell_k^\alpha(y) f(y)\eta_\alpha(dy) \\ &= \int_{\mathbb{R}_+^d} \frac{t}{\sqrt{4\pi}} \int_0^\infty \left(\sum_{k\in\mathbb{N}^d} e^{-s\left(|k|+\frac{|\alpha|+d}{2}\right)} \ell_k^\alpha(x)\ell_k^\alpha(y)\right) s^{-\frac{3}{2}} e^{-\frac{t^2}{4s}}\, ds f(y)\eta_\alpha(dy) \\ &= \int_{\mathbb{R}_+^d} P_t^\alpha(x,y) f(y)\eta_\alpha(dy), \end{aligned}$$

where

$$P_t^\alpha(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty G_s^\alpha(x,y) s^{-3/2} e^{-t^2/(4s)}\, ds.$$

We also have the subordination formula

$$f^\alpha(t,x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty g^\alpha(s,x) s^{-3/2} e^{-t^2/(4s)}\, ds, \quad t > 0. \tag{1.25}$$

Interchanging the order of integration and summation above is justified by (1.13) and Lemma 1.3.2.

PROPOSITION 1.3.10. *Let $1 \le p < \infty$, $\omega \in A_p^\alpha$ and $f \in L^p(\omega d\eta_\alpha)$. The Poisson integral $f^\alpha(t,x)$ of f is a C^∞ function on $\mathbb{R}_+ \times \mathbb{R}_+^d$. Moreover, it satisfies*

$$\left(L_x + \frac{\partial^2}{\partial t^2}\right) f^\alpha(t,x) = 0.$$

PROOF. Apply the arguments from the proof of Proposition 1.3.5. □

THEOREM 1.3.11. *Assume that $1 < p < \infty$, $\omega \in (A_p^\alpha)^*$ and $f \in L^p(\omega d\eta_\alpha)$. Let $f^\alpha(t,x)$ be the Poisson integral of f. Then*

(a) $\sup_{t>0} |f^\alpha(t,x)| \le CM_s^\alpha f(x), \quad x \in \mathbb{R}_+^d$;
(b) $\|f^\alpha(t,\cdot)\|_{L^p(\omega d\eta_\alpha)} \le C \exp\big(-t\sqrt{(|\alpha|+d)/2}\big)\|f\|_{L^p(\omega d\eta_\alpha)}$;
(c) $\|f^\alpha(t,\cdot) - f\|_{L^p(\omega d\eta_\alpha)} \longrightarrow 0, \quad t \to 0^+$;
(d) $f^\alpha(t,x) \longrightarrow f(x)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{P_t^\alpha\}_{t>0}$, $P_t^\alpha f(x) = f^\alpha(t,x)$, is a strongly continuous and uniformly bounded semigroup of operators on $L^p(\mathbb{R}_+^d, \omega d\eta_\alpha)$.

PROOF. Using the subordination formula (1.25) and (1.24) we get

$$\begin{aligned} |f^\alpha(t,x)| &\le CM_s^\alpha f(x) \int_0^\infty e^{-s(|\alpha|+d)/2} \frac{t}{\sqrt{4\pi}} s^{-3/2} e^{-t^2/(4s)}\, ds \\ &= Ce^{-t\sqrt{(|\alpha|+d)/2}} M_s^\alpha f(x). \end{aligned}$$

This shows (a) and (b). The rest of the proof is similar to the proof of Theorem 1.3.6. □

PROPOSITION 1.3.12. *Let $1 \le p, q < \infty$, $\omega \in A_p^\alpha$, $\rho \in A_q^\alpha$ and $f \in L^p(\omega d\eta_\alpha)$. Then*

$$\|P_t^\alpha f\|_{L^q(\rho d\eta_\alpha)} \le C(t)\|f\|_{L^p(\omega d\eta_\alpha)},$$

$C(t)$, $t > 0$, being a continuous and decreasing function of t that vanishes at infinity.

PROOF. Argue as in the proof of Proposition 1.3.8. □

COROLLARY 1.3.13. *Let $1 \le p < \infty$ and $\omega \in A_p^\alpha$. The family $\{P_t^\alpha\}_{t>0}$ is a strongly continuous semigroup of operators on $L^p(\mathbb{R}_+^d, \omega d\eta_\alpha)$.*

REMARK 1.3.14. A large part of the results of this section are valid for the space $L^\infty(\mathbb{R}_+^d)$. More precisely, Lemmas 1.3.1 and 1.3.2 and Propositions 1.3.5 and 1.3.10 remain valid if we replace $L^p(\omega d\eta_\alpha)$ by L^∞. Moreover, Theorems 1.3.6 and 1.3.11, except (c) and (d), also remain valid with L^∞ replacing $L^p(\omega d\eta_\alpha)$. Concerning (c) and (d), we have $\|g^\alpha(t,\cdot) - f\|_\infty \to 0$ and $\|f^\alpha(t,\cdot) - f\|_\infty \to 0$, $t \to 0^+$, but only for $f \in C_0(\mathbb{R}_+^d)$.

REMARK 1.3.15. If $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}_+^d, d\eta_\alpha)$ (the case $\omega \equiv 1$) then (b) of Theorem 1.3.6 holds with the coefficient $C\exp(-t(|\alpha|+d)/2)$ replaced by $(\cosh(t/2))^{-(|\alpha|+d)}$ (cf. computations in [**St**]) and Theorem 1.3.11 (b) holds with $C\exp\big(-t\sqrt{(|\alpha|+d)/2}\big)$ dropped. This means, in particular, that $\{T_t^\alpha\}_{t>0}$ and $\{P_t^\alpha\}_{t>0}$ are semigroups of contractions on $L^p(\mathbb{R}_+^d, d\eta_\alpha)$, $1 \le p \le \infty$. Note also that the above together with Lemma 1.3.3 implies L^p convergence (part (c) in both theorems) for $f \in L^p(d\eta_\alpha)$, $1 \le p < \infty$, which for $p = 1$ could not be concluded earlier.

REMARK 1.3.16. In dimension one we have $M_s^\alpha = M^\alpha$ (here M^α denotes the Hardy-Littlewood maximal function in $\mathbb{R}_+$ with respect to the measure η_α). Consequently, Theorems 1.3.6 (d) and 1.3.11 (d) hold with $p = 1$ admitted.

1.4. Laguerre function expansions; system $\{\varphi_k^\alpha\}$

Let $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in (-1, \infty)^d$ be multi-indices. The Laguerre function φ_k^α on $\mathbb{R}_+^d$ is defined as

$$\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}^{\alpha_d}(x_d), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}_+^d,$$

where $\varphi_{k_i}^{\alpha_i}$ are the one-dimensional Laguerre functions given by

$$\varphi_{k_i}^{\alpha_i}(x_i) = \left(\frac{2\Gamma(k_i + 1)}{\Gamma(k_i + \alpha_i + 1)} \right)^{1/2} L_{k_i}^{\alpha_i}(x_i^2) x_i^{\alpha_i + 1/2} e^{-x_i^2/2}, \qquad x_i > 0, \quad i = 1, \dots, d.$$

Note that only for $\alpha_i \geq -1/2$, $i = 1, \dots, d$, do the functions φ_k^α belong to all L^p-spaces on $\mathbb{R}_+^d$, $1 \leq p < \infty$. Therefore we assume throughout this section that $\alpha \in [-1/2, \infty)^d$.

Each φ_k^α is an eigenfunction of the differential operator

$$L = \Delta - |x|^2 - \sum_{i=1}^d \frac{1}{x_i^2} \left(\alpha_i^2 - \frac{1}{4} \right),$$

the corresponding eigenvalue being $-(4|k| + 2|\alpha| + 2d)$. The operator $-L$ is positive and symmetric in $L^2(\mathbb{R}_+^d, dx)$. Furthermore, the system $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}_+^d, dx)$.

The following estimate of φ_k^α is essential for our considerations:

$$|\varphi_k^\alpha(x)| \leq c \prod_{i=1}^d \Psi_{k_i}^{\alpha_i}(x_i), \quad x \in \mathbb{R}_+^d, \tag{1.26}$$

where

$$\Psi_{k_i}^{\alpha_i}(x_i) = \begin{cases} 1, & 0 < x_i \leq 4(2k_i + \alpha_i + 1); \\ \exp(-\gamma x_i), & x_i > 4(2k_i + \alpha_i + 1). \end{cases}$$

Here c and γ are independent of k and x. Similarly to (1.13), the above estimate follows by Muckenhoupt's generalization of the estimates proved by Askey and Wainger.

In this section we denote by $A_p = A_p(\mathbb{R}_+^d, dx)$, $1 \leq p < \infty$, the class of A_p weights on $\mathbb{R}_+^d$ with respect to the Lebesgue measure dx.

Let $1 \leq p < \infty$ and $\omega \in A_p$. The lemmas below are analogues of Lemmas 1.3.1–1.3.3. Their proofs are almost the same as for the system $\{\ell_k^\alpha\}$, the only essential difference being the estimate for Laguerre functions (1.26).

LEMMA 1.4.1. *There exists a constant c independent of $k \in \mathbb{N}^d$ such that*

$$\|\varphi_k^\alpha\|_{L^p(\omega)} \leq c(2|k| + |\alpha| + d)^d.$$

Moreover, the Fourier-Laguerre coefficients $\langle \varphi_k^\alpha, f \rangle = \int_{\mathbb{R}_+^d} \varphi_k^\alpha(x) f(x) dx$ exist for $f \in L^p(\omega)$ and they satisfy

$$|\langle \varphi_k^\alpha, f \rangle| \leq C(2|k| + |\alpha| + d)^d \|f\|_{L^p(\omega)},$$

with a constant C independent of $k \in \mathbb{N}^d$.

LEMMA 1.4.2. *The subspace spanned by* $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ *is dense in* $L^p(\omega)$ *and in* $C_0(\mathbb{R}_+^d)$ *with* $\|\cdot\|_\infty$ *norm.*

COROLLARY 1.4.3. *Let* $f \in L^p(\omega)$. *If* $\langle \varphi_k^\alpha, f\rangle = 0$ *for all* $k \in \mathbb{N}^d$ *then* $f = 0$.

Assume that $1 \le p < \infty$ and $\omega \in A_p$. Given $f \in L^p(\omega)$ we define its *heat-diffusion integral* by

$$g^\alpha(t,x) = \sum_{n=0}^{\infty} e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \langle \varphi_k^\alpha, f\rangle \varphi_k^\alpha(x), \quad t > 0.$$

The above series converges by (1.26) and Lemma 1.4.1. Similarly to the case of the system $\{\ell_k^\alpha\}$ we obtain an integral form of $g^\alpha(t,x)$:

$$\begin{aligned} g^\alpha(t,x) &= \int_{\mathbb{R}_+^d} \left(\sum_{n=0}^{\infty} e^{-t(4n+2|\alpha|+2d)} \sum_{|k|=n} \varphi_k^\alpha(x)\varphi_k^\alpha(y) \right) f(y)\, dy \\ &= \int_{\mathbb{R}_+^d} G_t^\alpha(x,y) f(y)\, dy. \end{aligned}$$

The kernel $G_t^\alpha(x,y)$ may computed by using the formula [**Le**, (4.17.6)] to be

(1.27)

$$G_t^\alpha(x,y) = (\sinh 2t)^{-d} \exp\left(-\frac{1}{2}\coth(2t)\left(|x|^2 + |y|^2\right)\right) \prod_{i=1}^{d} \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh(2t)}\right).$$

Note that $G_t^\alpha(x,y)$ is positive for $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$.

PROPOSITION 1.4.4. *Let* $1 \le p < \infty$, $\omega \in A_p$ *and* $f \in L^p(\omega)$. *The heat-diffusion integral* $g^\alpha(t,x)$ *of* f *is a* C^∞ *function on* $\mathbb{R}_+ \times \mathbb{R}_+^d$. *Moreover, it satisfies*

$$\left(L_x - \frac{\partial}{\partial t}\right) g^\alpha(t,x) = 0.$$

PROOF. The conclusion follows by the reasoning from the proof of Proposition 1.3.5 provided we have a proper estimate of $\partial^\beta \varphi_k^\alpha$ at our disposal. Since (cf. [**Le**, (4.18.6)])

$$\frac{\partial}{\partial x_j}\varphi_k^\alpha(x) = -2\sqrt{k_j}\varphi_{k-e_j}^{\alpha+e_j}(x) + \left(\frac{2\alpha_j + 1}{2x_j} - x_j\right)\varphi_k^\alpha(x), \quad k_j > 0,$$

the estimate (1.26) gives what is needed:

$$\left|\frac{\partial}{\partial x_j}\varphi_k^\alpha(x)\right| \le c\left(\varepsilon + \varepsilon^{-1}\right)\sqrt{|k|}, \quad x \in [\varepsilon, \varepsilon^{-1}]^d,$$

for any $\varepsilon \in (0,1)$; similarly for higher order derivatives. □

Denote by M_+ the (centered) Hardy-Littlewood maximal function in $\mathbb{R}_+^d$, i.e.

$$M_+ f(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|\, dy, \qquad x \in \mathbb{R}_+^d,$$

where the supremum is taken over all sets of the form $Q = \widetilde{Q} \cap \mathbb{R}_+^d$, and $\widetilde{Q}$ are cubes (with sides parallel to the coordinate axes) in $\mathbb{R}^d$ centered at x.

THEOREM 1.4.5. *Assume that* $1 \le p < \infty$, $\omega \in A_p$ *and* $f \in L^p(\omega)$. *Let* $g^\alpha(t,x)$ *be the heat-diffusion integral of* f. *Then*

(a) $|g^\alpha(t,x)| \le CM_+f(x), \quad x \in \mathbb{R}^d_+$;
(b) $\|g^\alpha(t,\cdot)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$;
(c) $\|g^\alpha(t,\cdot) - f\|_{L^p(\omega)} \longrightarrow 0, \quad t \to 0^+$;
(d) $g^\alpha(t,x) \longrightarrow f(x)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{T_t^\alpha\}_{t>0}$, $T_t^\alpha f(x) = g^\alpha(t,x)$, *is a strongly continuous and uniformly bounded semigroup of operators on* $L^p(\omega)$.

PROOF. Let $W(x) = (4\pi)^{-d/2}\exp(-|x|^2/4)$ be the Gauss-Weierstrass kernel in $\mathbb{R}^d$ and let $W_\varepsilon(\cdot) = \varepsilon^{-d}W(\cdot/\varepsilon)$ be its ε-dilation. We claim that there exists a constant C depending only on α such that

$$G_t^\alpha(x,y) \le C(2\pi)^{d/2} W_{\sinh(2t)/2}(y-x). \tag{1.28}$$

To prove this we need the following estimate (see [**Mu1**, p.238]) for I_β, $\beta \ge -1/2$:

$$c^{-1}\Xi(s) \le I_\beta(s) \le c\Xi(s), \qquad s>0, \tag{1.29}$$

with c depending only on β and the function Ξ defined by

$$\Xi(s) = \begin{cases} s^\beta & 0 < s \le 1, \\ s^{-1/2}e^s & 1 < s < \infty. \end{cases}$$

Let $u = (\sinh 2t)^{-1}$. Then we have (see 1.27)

$$\begin{aligned} G_t^\alpha(x,y) &= \prod_{i=1}^d u \exp\left(-\sqrt{u^2+1}(x_i^2+y_i^2)/2\right)\sqrt{x_iy_i}\, I_{\alpha_i}(ux_iy_i) \\ &= \prod_{i=1}^d \Upsilon_u^{\alpha_i}(x_i,y_i), \end{aligned}$$

and therefore to justify (1.28) it is sufficient to obtain the following bounds

$$\Upsilon_u^{\alpha_i}(x_i,y_i) \le c_{\alpha_i}\sqrt{u}\exp\left(-u(y_i-x_i)^2/2\right), \qquad i=1,\dots,d.$$

If $ux_iy_i \le 1$ then using (1.29) we get (recall that $\alpha_i \ge -1/2$)

$$\begin{aligned} \Upsilon_u^{\alpha_i}(x_i,y_i) &\le c_{\alpha_i}u^{\alpha_i+1}(x_iy_i)^{\alpha_i+1/2}\exp\left(-\sqrt{u^2+1}(x_i^2+y_i^2)/2\right) \\ &\le c_{\alpha_i}\sqrt{u}(ux_iy_i)^{\alpha_i+1/2}\exp\left(-u(x_i^2+y_i^2)/2\right) \\ &\le c_{\alpha_i}\sqrt{u}\exp\left(-u(x_i-y_i)^2/2\right). \end{aligned}$$

If $ux_iy_i > 1$ then, again by (1.29), we have

$$\begin{aligned} \Upsilon_u^{\alpha_i}(x_i,y_i) &\le c_{\alpha_i}\sqrt{u}\exp\left(-\sqrt{u^2+1}(x_i^2+y_i^2)/2 + ux_iy_i\right) \\ &\le c_{\alpha_i}\sqrt{u}\exp\left(-u(x_i^2+y_i^2-2x_iy_i)/2\right) \\ &= c_{\alpha_i}\sqrt{u}\exp\left(-u(x_i-y_i)^2/2\right). \end{aligned}$$

The claim is proved. Let $\widetilde{f}$ be an extension of f to $\mathbb{R}^d$ such that $\widetilde{f}(x)=0$ for $x \notin \mathbb{R}^d_+$. By (1.28) we have $|g^\alpha(t,x)| \le c|\widetilde{f}| * W_{\sinh(2t)/2}(x)$ and hence (a) and (b) follow by the $\mathbb{R}^d$ versions of Lemma 1.2.2 and Remark 1.2.3. This together with Lemma 1.4.2 justifies (c) and (d) in a standard manner.

The semigroup property is immediately verified for any φ_k^α, hence by (b) and Lemma 1.4.2 it holds for all $f \in L^p(\omega)$. Strong continuity follows by standard arguments, with the aid of (b) and Lemma 1.4.2. $\square$

PROPOSITION 1.4.6. *Let $1 \le p, q < \infty$, $\omega \in A_p$, $\rho \in A_q$ and $f \in L^p(\omega)$. Then*

$$\|T_t^\alpha f\|_{L^q(\rho)} \le C(t)\|f\|_{L^p(\omega)},$$

where $C(t)$, $t > 0$, is a continuous and decreasing function of t that vanishes at infinity.

PROOF. Apply Lemma 1.4.1 (see the proof of Proposition 1.3.8). □

Let us pass to Poisson integrals. Assume that $1 \le p < \infty$ and $\omega \in A_p$. Given $f \in L^p(\omega)$ we define its *Poisson integral* by

$$f^\alpha(t,x) = \sum_{n=0}^{\infty} e^{-t\sqrt{4n+2|\alpha|+2d}} \sum_{|k|=n} \langle \varphi_k^\alpha, f\rangle \varphi_k^\alpha(x), \quad t > 0.$$

The above series converges by (1.26) and Lemma 1.4.1. An integral form of $f^\alpha(t,x)$ as well as the corresponding subordination formula are obtained by (1.10). Applying arguments from the proof of Proposition 1.4.4 we get

PROPOSITION 1.4.7. *Let $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. The Poisson integral $f^\alpha(t,x)$ of f is a C^∞ function on $\mathbb{R}_+ \times \mathbb{R}_+^d$. Moreover, it satisfies*

$$\left(L_x + \frac{\partial^2}{\partial t^2}\right) f^\alpha(t,x) = 0.$$

The main result on Poisson integrals reads as follows.

THEOREM 1.4.8. *Assume that $1 \le p < \infty$, $\omega \in A_p$ and $f \in L^p(\omega)$. Let $f^\alpha(t,x)$ be the Poisson integral of f. Then*

(a) $|f^\alpha(t,x)| \le CM_+f(x), \quad x \in \mathbb{R}_+^d$;
(b) $\|f^\alpha(t,\cdot)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$;
(c) $\|f^\alpha(t,\cdot) - f\|_{L^p(\omega)} \longrightarrow 0, \quad t \to 0^+$;
(d) $f^\alpha(t,x) \longrightarrow f(x)$ *a.e.*, $\quad t \to 0^+$.

Moreover, the family $\{P_t^\alpha\}_{t>0}$, $P_t^\alpha f(x) = f^\alpha(t,x)$, is a strongly continuous and uniformly bounded semigroup of operators on $L^p(\omega)$.

PROOF. Items (a) and (b) follow by Theorem 1.4.5 and the subordination formula (see the proof of Theorem 1.2.12). The rest is justified as in the case of the heat-diffusion integrals. □

PROPOSITION 1.4.9. *Let $1 \le p, q < \infty$, $\omega \in A_p$, $\rho \in A_q$ and $f \in L^p(\omega)$. Then*

$$\|P_t^\alpha f\|_{L^q(\rho)} \le C(t)\|f\|_{L^p(\omega)},$$

where $C(t)$, $t > 0$, is a continuous and decreasing function of t that vanishes at infinity.

PROOF. Arguments are analogous to those from the proof of Proposition 1.3.8. □

REMARK 1.4.10. A large part of the results of this section are valid for the space $L^\infty(\mathbb{R}_+^d)$. More precisely, Lemma 1.4.1, Proposition 1.4.4 and Proposition 1.4.7 remain valid, if we replace $L^p(\omega)$ by L^∞. Further, Theorem 1.4.5 and Theorem 1.4.8, except (c) and (d), also remain valid with L^∞ replacing $L^p(\omega)$. Concerning (c) and (d), we have $\|g^\alpha(t,\cdot) - f\|_\infty \to 0$ and $\|f^\alpha(t,\cdot) - f\|_\infty \to 0$, $t \to 0^+$, but only for $f \in C_0(\mathbb{R}_+^d)$.

REMARK 1.4.11. Let M_s be the strong maximal function in $\mathbb{R}^d_+$ and denote by A_p^* the strong A_p class of weights in $\mathbb{R}^d_+$. Using [**St**, Lemma 4.2] and Lemma 1.3.7 one may obtain

$$|g^\alpha(t,x)| \le Ce^{-2t|\alpha\wedge 1/2|}M_s f(x), \quad x\in\mathbb{R}^d_+,$$

with the notation $|\alpha\wedge 1/2| = \sum_{i=1}^d \min(\alpha_i, 1/2)$. Thus, when $1<p<\infty$ and $\omega\in A_p^*$, the constants C in Theorem 1.4.5 (b) and Theorem 1.4.8 (b) may be replaced by $C\exp(-2t\,|\alpha\wedge 1/2|)$ and $C\exp(-t\sqrt{2\max\{|\alpha\wedge 1/2|\,,0\}})$, respectively. If $|\alpha\wedge 1/2|>0$ this gives the exponential decrease in t at infinity.

REMARK 1.4.12. If $1\le p\le\infty$ and $f\in L^p(\mathbb{R}^d_+)$ (the case $\omega\equiv 1$) then Theorem 1.4.5 (b) holds with C replaced by $C\exp(-2t(|\alpha|+d))$ (cf. estimates in [**St**]) and Theorem 1.4.8 (b) holds with $C\exp(-t\sqrt{2(|\alpha|+d)})$ instead of C.

1.5. Connection between Hermite and Laguerre function expansions

In this section we briefly show how some results for Hermite semigroups may be transferred to the Laguerre setting. We will exploit the ideas used by Dinger [**Di**] and developed later in [**GIT**], for Hermite and Laguerre polynomial systems (see also Chapter 3). The lemmas we shall use are straightforward modifications of those in [**GIT**] and therefore we provide no proofs here.

The key fact underlying the idea of transference is that if α has a special, half-integer form, then the one-dimensional Laguerre functions ℓ_k^α can be expressed in a suitable way by means of multi-dimensional Hermite functions. The following lemma makes this precise.

LEMMA 1.5.1. *Let ℓ_k^α be the one-dimensional Laguerre function of type α with $\alpha=n/2-1$, $n\in\mathbb{N}\backslash\{0\}$, and let $x\in\mathbb{R}^n$. Then we have the expansion*

$$\ell_k^\alpha(|x|^2)=\sum_{|r|=k} a_r h_{2r}(x), \qquad r=(r_1,\dots,r_n)\in\mathbb{N}^n.$$

Now, let $n=(n_1,\dots,n_d)\in\mathbb{N}^d$ be a multi-index and define $x^i=(x_1^i,\dots,x_{n_i}^i)\in\mathbb{R}^{n_i}$, $i=1,\dots,d$. We define the quadratic transformation $\phi\colon\mathbb{R}^{|n|}\longrightarrow\mathbb{R}^d$ by

$$\phi(x^1,\dots,x^d)=\left(|x^1|^2,\dots,|x^d|^2\right). \tag{1.30}$$

LEMMA 1.5.2. *Let $\alpha=(\alpha_1,\dots,\alpha_d)$ with $\alpha_i=n_i/2-1$ and $n\in(\mathbb{N}\backslash\{0\})^d$. Given a weight ω in $\mathbb{R}^d_+$ and a measurable function f, the following holds:*

$$c_{d,n}\int_{\mathbb{R}^d_+} f(y)\omega(y)y^\alpha dy=\int_{\mathbb{R}^{|n|}} f\circ\phi(x)\;\omega\circ\phi(x)\,dx,$$

provided one of the integrals is absolutely convergent.

Recall that $\eta_\alpha(dx)=x^\alpha dx$.

LEMMA 1.5.3. *Let α, n and ω be as in Lemma 1.5.2, $p\in[1,\infty)$, and let f be a fixed function in $L^p(\mathbb{R}^d_+,\omega d\eta_\alpha)$. Suppose that $T,\widetilde{T}$ are operators defined on $L^p(\mathbb{R}^d_+,\omega d\eta_\alpha)$ and $L^p(\mathbb{R}^{|n|},\omega\circ\phi)$ respectively, satisfying $(Tf)(\phi(x))=\widetilde{T}(f\circ\phi)(x)$ for $x\in\mathbb{R}^{|n|}$. If*

$$\|\widetilde{T}f\|_{L^p(\mathbb{R}^{|n|},\omega\circ\phi)}\le C\|f\|_{L^p(\mathbb{R}^{|n|},\omega\circ\phi)}$$

then also

$$\|Tf\|_{L^p(\mathbb{R}^d_+,\omega d\eta_\alpha)}\le C\|f\|_{L^p(\mathbb{R}^d_+,\omega d\eta_\alpha)}$$

with the same constant C. Moreover, in the case $p=1$ an analogous statement is true for weighted weak type inequalities.

Let $\{T_t^\alpha\}$ and $\{P_t^\alpha\}$ be the heat-diffusion and Poisson semigroups associated with the Laguerre system $\{\ell_k^\alpha\}$. Denote by $\{T_t^H\}$ and $\{P_t^H\}$ the corresponding semigroups for the system of Hermite functions $\{h_k\}$ (see [**Th3**] for basic facts concerning this system).

LEMMA 1.5.4. *Assume that α and n are as in Lemma 1.5.2. Then for any $f \in \operatorname{lin}\{\ell_k^\alpha : k \in \mathbb{N}^d\}$ we have*

$$(T_t^\alpha f)\circ\phi = T_{t/2}^H(f\circ\phi), \qquad (P_t^\alpha f)\circ\phi = P_{t/\sqrt{2}}^H(f\circ\phi).$$

Let $A_p^\alpha = A_p(\mathbb{R}_+^d, d\eta_\alpha)$ be the class of weights from Section 1.3. For α and n as in Lemma 1.5.2 we define

$$\widetilde{A}_p^\alpha = \{\omega \in A_p^\alpha : \omega\circ\phi \in A_p(\mathbb{R}^{|n|})\}.$$

The above class is considerably large. In fact, the following inclusion holds:

$$(A_p^\alpha)^* \subset \widetilde{A}_p^\alpha, \quad 1 \le p < \infty. \tag{1.31}$$

We shall sketch a proof of this fact for $p>1$ (the same reasoning applies if $p=1$).

Let $Q \subset \mathbb{R}^{|n|}$ be a cube with sides parallel to the coordinate axes. We have $Q = Q_1\times\cdots\times Q_d$, where each Q_i is a cube in $\mathbb{R}^{n_i}$. Denote by S_i the smallest rectangle in polar coordinates in $\mathbb{R}^{n_i}$ that contains Q_i. Note that $|S_i|$ and $|Q_i|$ are comparable with a constant independent of Q_i. Thus so are $|Q|$ and $|S|$, $S = S_1\times\cdots\times S_d$. Given a weight function ω in $\mathbb{R}^d$ we get

$$\frac{1}{|Q|}\int_Q \omega\circ\phi(x)\,dx \le c\frac{1}{|S|}\int_S \omega\circ\phi(x)\,dx = c\frac{1}{|\widetilde{S}|}\int_{\widetilde{S}} \omega\circ\phi(x)\,dx. \tag{1.32}$$

Here $\widetilde{S} = \widetilde{S}_1\times\cdots\times\widetilde{S}_d$, with $\widetilde{S}_i$ being the radialization of S_i in $\mathbb{R}^{n_i}$. The last equality in (1.32) holds since $\omega\circ\phi$ is poly-radial on $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_d}$. Now, making proper change of variables and integrating in polar coordinates on each $\mathbb{R}^{n_i}$ we obtain

$$\frac{1}{|\widetilde{S}|}\int_{\widetilde{S}} \omega\circ\phi(x)\,dx = c\frac{1}{\eta_\alpha(\phi(\widetilde{S}))}\int_{\phi(\widetilde{S})} \omega(y)\eta_\alpha(dy).$$

Treating similarly $(\omega\circ\phi)^{-p'/p}$ we conclude that

$$\left[\frac{1}{|Q|}\int_Q \omega\circ\phi(x)\,dx\right]\left[\frac{1}{|Q|}\int_Q (\omega\circ\phi(x))^{-p'/p}\,dx\right]^{p/p'}$$
$$\le c\left[\frac{1}{\eta_\alpha(\phi(\widetilde{S}))}\int_{\phi(\widetilde{S})} \omega(y)\eta_\alpha(dy)\right]\left[\frac{1}{\eta_\alpha(\phi(\widetilde{S}))}\int_{\phi(\widetilde{S})} (\omega(y))^{-p'/p}\eta_\alpha(dy)\right]^{p/p'},$$

with c independent of Q. Since $\phi(\widetilde{S})$ is a rectangle in $\mathbb{R}^d$, this clearly shows that $\omega\circ\phi \in A_p(\mathbb{R}^{|n|})$ if only $\omega \in (A_p^\alpha)^*$. Hence (1.31) follows.

As a corollary of the above lemmas, Lemma 1.3.3 and the results for Hermite semigroups [**StTo**, Theorems 2.6 and 2.8], we obtain

THEOREM 1.5.5. *Let $\alpha = (\alpha_1,\dots,\alpha_d)$ with $\alpha_i = n_i/2-1$ and $n_i \in \mathbb{N}\backslash\{0\}$. Assume that $\omega \in \widetilde{A}_p^\alpha$. Then the maximal operators*

$$T_t^* f = \sup_{t>0}|T_t^\alpha f| \quad \textit{and} \quad P_t^* f = \sup_{t>0}|P_t^\alpha f|$$

defined on $L^p(\omega d\eta_\alpha)$ *are bounded if* $1 < p < \infty$, *and weakly bounded if* $p = 1$. *Moreover, the semigroups* $\{T_t^\alpha\}$ *and* $\{P_t^\alpha\}$ *are uniformly bounded on* $L^p(\omega d\eta_\alpha)$, $1 \leq p < \infty$.

As a consequence of Theorem 1.5.5 and Lemma 1.3.3 we get

COROLLARY 1.5.6. *Let* $1 \leq p < \infty$ *and* $f \in L^p(\omega d\eta_\alpha)$. *Under the assumptions of Theorem 1.5.5, we have*

$$T_t^\alpha f \longrightarrow f \quad \text{and} \quad P_t^\alpha f \longrightarrow f, \quad t \longrightarrow 0^+,$$

the convergence being both in $L^p(\omega d\eta_\alpha)$ *and almost everywhere.*

REMARK 1.5.7. When the multi-index α is half-integer, Theorem 1.5.5 extends Theorems 1.3.6 and 1.3.11. This generalization is particularly significant in the case $p = 1$, since weak boundedness of T_t^* and P_t^* seems to be a new result even in the unweighted setting ($\omega \equiv 1$) if $d > 1$.

REMARK 1.5.8. Still another transference may be carried out from the special Hermite setting to the one-dimensional Laguerre setting based on the system $\{\psi_k^\alpha : k \in \mathbb{N}\}$, $\alpha > -1$. The functions ψ_k^α are defined by

$$\psi_k^\alpha(r) = \left(\frac{\Gamma(k+1)}{2^\alpha \Gamma(k+\alpha+1)}\right)^{1/2} L_k^\alpha\left(\frac{r^2}{2}\right) \exp\left(-\frac{1}{4}r^2\right) = 2^{-\alpha/2} \ell_k^\alpha\left(\frac{r^2}{2}\right), \qquad r > 0,$$

and they constitute an orthonormal basis in $L^2(\mathbb{R}_+, r^{2\alpha+1}dr)$. If $f(z) = \tilde{f}(r)$, $r = |z|$, is a radial function on $\mathbb{C}^n$, then its special Hermite expansion reduces to the Laguerre expansion of $\tilde{f}$ with respect to the system $\{\psi_k^{n-1}\}$ (see [**Th4**] for details). Using the results of Section 1.2 and the fact that if $\omega \in A_p(\mathbb{R}_+, r^{2n-1}dr)$ then $\omega(|\cdot|) \in A_p(\mathbb{C}^n)$, which is justified similarly to (1.31), we obtain for $\{\psi_k^{n-1}\}$ conclusions analogous to those from Theorem 1.5.5.

REMARK 1.5.9. To treat the system $\{\psi_k^\alpha\}$ also in higher dimensions and for all half-integer multi-indices α one may use the transference from Hermite function expansions, but with the quadratic transformation (1.30) replaced by $\phi(x^1, \dots, x^d) = (|x^1|, \dots, |x^d|)$. In particular an analogue of Theorem 1.5.5 follows.

REMARK 1.5.10. Similar analysis to those from Sections 1.3 and 1.4 may be conducted for another Laguerre system $\{\mathscr{L}_k^\alpha : k \in \mathbb{N}^d\}$, defined by

$$\mathscr{L}_k^\alpha(x) = \ell_k^\alpha(x) x^{\alpha/2}, \quad x \in \mathbb{R}_+^d.$$

The system $\{\mathscr{L}_k^\alpha\}$ is an orthonormal basis in $L^2(\mathbb{R}_+^d, dx)$ and was investigated in one-dimensional, unweighted case by Stempak [**St**].

CHAPTER 2

Riesz transforms for polynomial Laguerre expansions

2.1. Introduction

The aim of this chapter is to study the Riesz transform $\mathcal{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ naturally associated with multi-dimensional Laguerre polynomial expansions of type α. Our main result is contained in Theorem 2.5.1: we prove that if $\alpha \in [-1/2, \infty)^d$ then R_j^α, $j = 1, \dots, d$, are bounded operators in L^p with appropriate measure for $1 < p < \infty$. Moreover, the corresponding L^p constants are independent of the dimension d and the type multi-index α. As a consequence we obtain boundedness and convergence results for the corresponding conjugate Poisson integrals, see Corollary 2.5.3 below.

Our methods are analytic and based on the Littlewood-Paley-Stein theory contained in the monograph [**S1**]. We construct appropriate square functions that relate a function and its Riesz transform, and then prove that these square functions are bounded in L^p, $1 < p < \infty$. Noteworthy, the same scheme was exploited by Gutiérrez [**Gu**], who considered Riesz transforms associated with the multi-dimensional Hermite semigroup. Nevertheless, the case of Laguerre semigroup is more involved.

Riesz transforms and conjugate Poisson integrals for the Laguerre semigroup were first studied by Muckenhoupt [**Mu3**]. However, he worked in the one-dimensional setting and methods he used seem to be inapplicable in higher dimensions. Recently a g-function and Riesz transforms associated with the multi-dimensional Laguerre semigroup were studied by Gutiérrez, Incognito and Torrea, [**GIT**]. The technique of "transference" exploited there allowed to obtain $L^p, 1 < p < \infty$, boundedness results only for a discrete set of half-integer multi-indices α. Here we remove this restriction and consider all intermediate multi-indices α. The case when $\alpha_i < -1/2$ for some $i = 1, \dots, d$ seems to require more subtle analysis and the corresponding Riesz transforms are not considered.

The chapter is organized as follows. Section 2.2 contains basic facts about Laguerre semigroups and related objects. In Section 2.3 we introduce and briefly study a family of modified Laguerre semigroups. These are of great importance in Section 2.4, where we define suitable square functions and prove necessary L^p inequalities. Main results of this section are contained in Theorems 2.4.1 and 2.4.2. Finally, in Section 2.5 we conclude the results concerning Riesz transforms and conjugate Poisson integrals.

2.2. Laguerre semigroups and related objects

Consider the measure μ_α in $\mathbb{R}_+^d$ given by $d\mu_\alpha(x) = \prod_{i=1}^d x_i^{\alpha_i} e^{-x_i}\, dx$. The *Laguerre differential operator*

$$\mathcal{L}^\alpha = -\sum_{i=1}^d \left[x_i \partial_{x_i}^2 + (\alpha_i + 1 - x_i)\partial_{x_i} \right]$$

is positive and symmetric in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Moreover, $\mathcal{L}^\alpha$ has a closure which is self-adjoint in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ and which also will be denoted by $\mathcal{L}^\alpha$. Each Laguerre polynomial L_k^α is an eigenfunction of $\mathcal{L}^\alpha$ with the corresponding eigenvalue $|k|$. Furthermore, the system $\{L_k^\alpha : k \in \mathbb{N}^d\}$ constitutes an orthogonal basis in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. Thus we have an orthogonal decomposition

$$L^2(\mathbb{R}^d_+, d\mu_\alpha) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_n = \operatorname{lin}\{L_k^\alpha : |k| = n\}$.

The semigroup generated by $\mathcal{L}^\alpha$ is called *Laguerre semigroup* and will be denoted by T_t^α. Given $f \in L^2(\mathbb{R}^d_+, d\mu_\alpha)$, it has the expansion

$$f = \sum_{k\in\mathbb{N}^d} a_k(f) L_k^\alpha,$$

where, with the notation $\langle f, L_k^\alpha\rangle_{d\mu_\alpha} = \int_{\mathbb{R}^d_+} f(y) L_k^\alpha(y) d\mu_\alpha(y)$,

$$a_k(f) = \langle f, L_k^\alpha\rangle_{d\mu_\alpha} / \|L_k^\alpha\|^2_{L^2(d\mu_\alpha)} = \langle f, L_k^\alpha\rangle_{d\mu_\alpha} \prod_{i=1}^{d} \frac{\Gamma(k_i+1)}{\Gamma(k_i+\alpha_i+1)},$$

hence

$$T_t^\alpha f = \sum_{k\in\mathbb{N}^d} a_k(f) e^{-t|k|} L_k^\alpha,$$

both series being convergent in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$. However, the definition of $T_t^\alpha f$ for $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$, $1 \le p \le \infty$, by means of its Fourier-Laguerre expansion would be unsatisfactory, since the corresponding series may diverge for $p < 2$, see [**Mu1**]. Therefore it is appropriate to use an integral definition, which turns out to be usable for all $p \in [1, \infty]$. We define

$$T_t^\alpha f(x) = \int_{\mathbb{R}^d_+} G_t^\alpha(x,y) f(y)\, d\mu_\alpha(y), \qquad f \in L^p(\mathbb{R}^d_+, d\mu_\alpha), \tag{2.1}$$

where

$$G_t^\alpha(x,y) = \sum_{k\in\mathbb{N}^d} e^{-t|k|} L_k^\alpha(x) L_k^\alpha(y) / \|L_k^\alpha\|^2_{L^2(d\mu_\alpha)}.$$

The kernel $G^\alpha(x,y)$ may be computed explicitly by means of the Hille-Hardy formula [**Le**, 4.17.6]. The result is

$$G_t^\alpha(x,y) = \prod_{j=1}^{d} (1-u)^{-1} \exp\left(-\frac{u}{1-u}(x_j+y_j)\right) \sqrt{ux_jy_j}^{\,-\alpha_j} I_{\alpha_j}\left(\frac{2\sqrt{ux_jy_j}}{1-u}\right),$$

where $u = e^{-t}$ and I_ν denotes the modified Bessel function of the first kind and order ν, cf. [**Le**]. Noteworthy, $G_t^\alpha(x,y)$ is smooth and strictly positive for $(t,x,y) \in \mathbb{R}^{2d+1}_+$. By using the estimate (1.29) for I_ν it is easily seen that the integral in (2.1) is absolutely convergent. It is well-known that $\{T_t^\alpha\}$ is a *symmetric diffusion semigroup* (in fact $\{T_t^\alpha\}$ is a transition semigroup for the Laguerre diffusion process, which already received an attention due to some applications in financial mathematics). In particular $T_t^\alpha \mathbf{1} = \mathbf{1}$ and

$$\|T_t^\alpha f\|_{L^p(d\mu_\alpha)} \le \|f\|_{L^p(d\mu_\alpha)}, \qquad 1 \le p \le \infty.$$

The corresponding Poisson semigroup $\{P_t^\alpha\}$ is defined by means of the subordination principle as the weighted average of T_t^α:

$$P_t^\alpha f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^\alpha f(x)\, du, \qquad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha).$$

Note, that for $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ we have

$$P_t^\alpha f = \sum_{k\in\mathbb{N}^d} a_k(f) e^{-t|k|^{1/2}} L_k^\alpha.$$

By a general theory (see [**S1**, p.73]) it follows that the maximal operator $(P^\alpha)^* f(x) = \sup_{t>0} |P_t^\alpha f(x)|$ satisfies

$$\left\|(P^\alpha)^* f\right\|_{L^p(d\mu_\alpha)} \le C_p \left\|f\right\|_{L^p(d\mu_\alpha)}, \qquad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha). \tag{2.2}$$

Let us emphasize that the constant C_p depends neither on the dimension d nor on the type multi-index α. The important consequence of (2.2) is

$$\lim_{t\to 0^+} P_t^\alpha f(x) = f(x) \ \ a.e., \qquad f \in L^p(\mathbb{R}_+^d, d\mu_\alpha), \quad 1 < p < \infty,$$

the fact which will be used later without further mention.

We define the i-th partial derivative associated with $\mathcal{L}^\alpha$ by

$$\delta_i = \sqrt{x_i}\partial_{x_i},$$

see [**GIT**]. The formal adjoint of δ_i in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ is given by

$$\delta_i^* = -\sqrt{x_i}\left(\partial_{x_i} + \frac{\alpha_i + 1/2 - x_i}{x_i}\right)$$

and we have

$$\mathcal{L}^\alpha = \sum_{i=1}^d \delta_i^* \delta_i.$$

The last equality may be written in a compact form

$$\mathcal{L}^\alpha = \operatorname{div}_\alpha \operatorname{grad}_\alpha,$$

where $\operatorname{grad}_\alpha = (\delta_1, \dots, \delta_d)$ and $\operatorname{div}_\alpha F = \sum_{i=1}^d \delta_i^* f_i$ for a vector-valued function $F(y) = (f_1(y), \dots, f_d(y))$.

The Riesz-Laguerre transform $\mathcal{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ is then formally defined by (cf. [**GIT**])

$$\mathcal{R}^\alpha = \operatorname{grad}_\alpha (\mathcal{L}^\alpha)^{-1/2}\, \Pi_0, \tag{2.3}$$

where Π_0 denotes the orthogonal projection onto the orthogonal complement $\mathcal{H}_0^\perp$ of the eigenspace corresponding to the Laguerre eigenvalue 0. Note, that (2.3) makes sense for Laguerre polynomials (hence for all polynomials) and by (2.4) we have

$$R_i^\alpha L_k^\alpha = -|k|^{-1/2}\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \qquad |k| > 0,$$

and $R_i^\alpha L_k^\alpha = 0$ if $|k| = 0$. This follows by the differentiation rule [**Le**, (4.18.6)]

$$\partial_{x_i} L_k^\alpha(x) = -L_{k-e_i}^{\alpha+e_i}(x), \qquad i = 1, \dots, d, \tag{2.4}$$

e_i denoting the i-th coordinate versor in $\mathbb{R}^d$. Here and later on we use the convention that $L_{k-e_i}^{\alpha+e_i} = 0$ if $k_i - 1 < 0$.

A crucial observation which should be made here is that $R_i^\alpha L_k^\alpha$ is not a Laguerre polynomial of the same type α (to make it worse, it is not a polynomial at all), which is a consequence of "bad" action of the Laguerre derivatives δ_i on L_k^α. This effect is absent in the Hermite setting and makes the present analysis more complex, including working with d auxiliary orthogonal systems and considering supplementary semigroups, which are not simply "translations" of P_t^α (as it was in the Hermite case, see [**Gu**]).

2.3. Modified Laguerre semigroups

Now, we introduce additional semigroups $\{\widetilde{P}_t^{\alpha,i}\}$, $i = 1, \dots, d$, which are generated by "slight" modifications of the operator $(\mathcal{L}^\alpha)^{1/2}$. As we shall see, they play an essential role in the study of Riesz transforms and conjugacy for Laguerre expansions. To proceed, we first define the operators

$$M_i^\alpha = \delta_i\delta_i^* + \sum_{j\neq i}\delta_j^*\delta_j \;=\; \mathcal{L}^\alpha + \frac{\alpha_i + 1/2 + x_i}{2x_i}, \qquad i = 1, \dots, d.$$

Observe that each M_i^α is symmetric, and positive since for sufficiently regular functions f

$$\langle M_i^\alpha f, f\rangle_{d\mu_\alpha} = \int \Big((\delta_i^* f)^2 + \sum_{j\neq i} (\delta_j f)^2 \Big) d\mu_\alpha.$$

We set (formally)

$$\widetilde{P}_t^{\alpha,i} = e^{-t(M_i^\alpha)^{1/2}}, \qquad i = 1, \dots, d.$$

To make this definition more explicit we will need the following

LEMMA 2.3.1. *Given $i = 1, \dots, d$ the functions $\sqrt{x_i}L_{k-e_i}^{\alpha+e_i}(x)$ are eigenfunctions of M_i^α, with the corresponding eigenvalues $|k|$. Moreover, the system $\{\sqrt{x_i}L_k^{\alpha+e_i}(x) : k \in \mathbb{N}^d\}$ forms an orthogonal basis in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$.*

PROOF. The first part follows by a direct computation and using the identity

$$\mathcal{L}^{\alpha+e_i} L_{k-e_i}^{\alpha+e_i} = (|k| - 1) L_{k-e_i}^{\alpha+e_i}.$$

The second part is a consequence of the fact that the system $\{L_k^{\alpha+e_i} : k \in \mathbb{N}^d\}$ is an orthogonal basis in $L^2(\mathbb{R}_+^d, d\mu_{\alpha+e_i})$. □

By the above lemma, given $i = 1, \dots, d$, any $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ has the expansion

$$f = \sum_{k\in\mathbb{N}^d} a_k^i(f)\sqrt{x_i}L_k^{\alpha+e_i},$$

hence

$$\widetilde{P}_t^{\alpha,i} f = \sum_{k\in\mathbb{N}^d} a_k^i(f) e^{-t(|k|+1)^{1/2}} \sqrt{x_i}L_k^{\alpha+e_i}. \tag{2.5}$$

Unfortunately, such definition is not appropriate for general $f \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$, $1 \le p \le \infty$, the reason for that being similar as in the case of T_t^α. Therefore we will use an integral representation. Let $f \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$ and define

$$\widetilde{T}_t^{\alpha,i} f(x) = \int_{\mathbb{R}_+^d} \widetilde{G}_t^{\alpha,i}(x,y) f(y)\, d\mu_\alpha(y), \qquad i = 1, \dots, d, \tag{2.6}$$

where

$$\widetilde{G}_t^{\alpha,i}(x,y) \;=\; \sum_{k\in\mathbb{N}^d} e^{-t(|k|+1)}\sqrt{x_i}L_k^{\alpha+e_i}(x)\sqrt{y_i}L_k^{\alpha+e_i}(y)/\|\sqrt{y_i}L_k^{\alpha+e_i}\|_{L^2(d\mu_\alpha)}^2$$

$$= e^{-t}\sqrt{x_i y_i}\sum_{k\in\mathbb{N}^d} e^{-t|k|} L_k^{\alpha+e_i}(x) L_k^{\alpha+e_i}(y)/\|L_k^{\alpha+e_i}\|^2_{L^2(d\mu_{\alpha+e_i})}$$
$$= e^{-t}\sqrt{x_i y_i}\, G_t^{\alpha+e_i}(x,y).$$

Similarly as in (2.1), the integral in (2.6) is absolutely convergent and for $f \in L^2(\mathbb{R}^d_+, d\mu_\alpha)$ we have

$$\widetilde{T}_t^{\alpha,i} f = \sum_{k\in\mathbb{N}^d} a_k^i(f) e^{-t(|k|+1)} \sqrt{x_i} L_k^{\alpha+e_i}. \tag{2.7}$$

Now, using the subordination formula, we set

$$\widetilde{P}_t^{\alpha,i} f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \widetilde{T}^{\alpha,i}_{t^2/(4u)} f(x)\, du, \qquad i = 1,\dots,d.$$

Note, that the above definition makes sense for $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$, $1 \le p \le \infty$, and if $p = 2$ it coincides with (2.5). Observe also, that the operators $\widetilde{P}_t^{\alpha,i}$ are positive and symmetric by the corresponding properties of $\widetilde{G}_t^{\alpha,i}(x,y)$. To obtain the relevant L^p inequalities for $\widetilde{P}_t^{\alpha,i}$ we prove the following.

LEMMA 2.3.2. *Given $i = 1,\dots,d$ the quantity*

$$\inf\left\{\lambda : \widetilde{G}_t^{\alpha,i}(x,y) \le \lambda e^{-t/2} G_t^\alpha(x,y),\ x,y \in \mathbb{R}^d_+,\ t > 0\right\}$$

is the one-dimensional, finite function of α_i, which is further denoted by Λ.

The function $\Lambda(\nu)$ is nonincreasing for $\nu \in (-1,\infty)$, $\Lambda(\nu) = 1$ for $\nu \ge -1/2$, $\Lambda(\nu) > 1$ for $\nu \in (-1,-1/2)$ and $\Lambda(\nu) \longrightarrow \infty$ as $\nu \longrightarrow -1^+$.

PROOF. By the explicit formula for $G_t^\alpha(x,y)$ we get

$$\widetilde{G}_t^{\alpha,i}(x,y) = \Theta_{\alpha_i}\left(\frac{2\sqrt{u x_j y_j}}{1-u}\right) e^{-t/2} G_t^\alpha(x,y),$$

where the function Θ_ν is given by

$$\Theta_\nu(z) = \frac{I_{\nu+1}(z)}{I_\nu(z)}, \qquad z > 0.$$

Quotients of this type are of independent interest and have been studied by many authors. In particular, it is known (see [**IfSi**]) that $\Theta_\nu(z)$ is a decreasing function of $\nu > -1$ for every fixed $z > 0$. Hence

$$\Lambda(\nu) = \sup_{z>0} \Theta_\nu(z)$$

is nonincreasing in the interval $(-1,\infty)$. Consequently, we have

$$\Lambda(\nu) \le \Lambda(-1/2) = \sup_{z>0} \tanh(z) = 1, \qquad \nu \ge -1/2.$$

Thus we obtain $\Lambda(\nu) = 1$ for $\nu \ge -1/2$, because (cf. [**Le**, 5.11.10])

$$I_\nu(z) = (2\pi z)^{-1/2} e^z [1 + O(1/z)], \quad z \longrightarrow \infty.$$

To treat $\Lambda(\nu)$ in the range $-1 < \nu < -1/2$ we shall use the estimate

$$B_\nu(z) \le \Theta_\nu(z), \qquad z > 0, \quad \nu > -1,$$

where

$$B_\nu(z) = \frac{z}{\nu + 1/2 + \sqrt{z^2 + (\nu+3/2)^2}}.$$

For this and other bounds for Θ_ν see [**Na**] and references therein. A simple calculus shows that

$$\sup_{z>0} B_\nu(z) = \frac{\nu+3/2}{\sqrt{2(\nu+1)}}, \qquad -1<\nu<-1/2,$$

hence $\Lambda(\nu)>1$ for $\nu\in(-1,-1/2)$, and $\Lambda(\nu)\longrightarrow\infty$ as $\nu\longrightarrow -1^+$.

Let us note, that more precise description of the behavior of Λ is possible by using another estimates for $\Theta_\nu(z)$, cf. [**Na**]. For example one can show that

$$\Lambda(\nu) = O\big((\nu+1)^{-1/2}\big), \qquad \nu\longrightarrow -1^+.$$

□

COROLLARY 2.3.3. *Given $i=1,\dots,d$ and a reasonable f we have*

$$\widetilde{P}_t^{\alpha,i}f(x)\le \Lambda(\alpha_i)P_t^\alpha f(x), \qquad x\in\mathbb{R}_+^d, \quad t>0.$$

Thus, $\{\widetilde{P}_t^{\alpha,i}\}$ is a uniformly bounded semigroup of operators (contractions, if $\alpha\in[-1/2,\infty)^d$) in $L^p(\mathbb{R}_+^d,d\mu_\alpha)$, $1\le p\le\infty$.

PROPOSITION 2.3.4. *Let $f\in L^2(\mathbb{R}_+^d,d\mu_\alpha)$ and $i\in\{1,\dots,d\}$. Then $T_t^\alpha f(x)$, $P_t^\alpha f(x)$, $\widetilde{T}_t^{\alpha,i}f(x)$, $\widetilde{P}_t^{\alpha,i}f(x)$ are C^∞ functions on $\mathbb{R}_+\times\mathbb{R}_+^d$. Moreover,*

$$(\partial_t+\mathcal{L}^\alpha)\,T_t^\alpha f(x) \;=\; 0 \;=\; (\partial_t+M_i^\alpha)\,\widetilde{T}_t^{\alpha,i}f(x), \qquad x\in\mathbb{R}_+^d,$$

$$\left(\partial_t^2-\mathcal{L}^\alpha\right)P_t^\alpha f(x) \;=\; 0 \;=\; \left(\partial_t^2-M_i^\alpha\right)\widetilde{P}_t^{\alpha,i}f(x), \qquad x\in\mathbb{R}_+^d.$$

PROOF. We consider only $\widetilde{T}_t^{\alpha,i}f(x)$ since treatment of the remaining functions is analogous. We will show that the series in (2.7) may be differentiated term by term. Observe that by Schwarz' inequality

$$|a_k^i(f)| \le \frac{\|f\|_{L^2(d\mu_\alpha)}}{\|\sqrt{y_i}L_k^{\alpha+e_i}\|_{L^2(d\mu_\alpha)}} = \frac{\|f\|_{L^2(d\mu_\alpha)}}{\sqrt{k_i+\alpha_i+1}}\prod_{j=1}^d\left(\frac{\Gamma(k_j+1)}{\Gamma(k_j+\alpha_j+1)}\right)^{1/2},$$

hence $|a_k^i(f)|$ grows at most polynomially in $|k|$. Furthermore, for a fixed compact set $K\subset\mathbb{R}_+^d$ the quantity $\sup_{x\in K}|L_k^{\alpha+e_i}(x)|$ also has sub-polynomial growth in $|k|$, see [**Mu3**, p.405]. Therefore the series defining $\widetilde{T}_t^{\alpha,i}f(x)$ may be differentiated in t term by term. The result is

$$\partial_t^m\widetilde{T}_t^{\alpha,i}f(x) = \sum_{k\in\mathbb{N}^d} a_k^i(f)(-1)^m(|k|+1)^m e^{-t(|k|+1)}\sqrt{x_i}L_k^{\alpha+e_i}(x), \tag{2.8}$$

the right hand side being continuous since the series converges almost uniformly in (t,x). Using (2.4) we see that also $\sup_{x\in K}|\partial_{x_j}(\sqrt{x_i}L_k^{\alpha+e_i}(x))|$ grows in $|k|$ not faster than polynomially and hence we may differentiate in x_j term by term the series in (2.8), the result being a continuous function since the convergence is again almost uniform in (t,x). The same arguments apply to higher derivatives, so $\widetilde{T}_t^{\alpha,i}f(x)$ is smooth on $\mathbb{R}_+\times\mathbb{R}_+^d$. The corresponding heat equation is easily verified by differentiating term by term the series of $\widetilde{T}_t^{\alpha,i}f(x)$. □

REMARK 2.3.5. Noteworthy, Proposition 2.3.4 is true for $f\in L^p(\mathbb{R}_+^d,d\mu_\alpha)$, $1\le p\le\infty$, which is proved by an analysis of the integral representations of the semigroups.

2.4. Littlewood-Paley type square functions

Denote $\nabla_\alpha = (\partial_t, \mathrm{grad}_\alpha)$. We consider the following Littlewood-Paley-Stein type square functions:

$$g(f)(x) = \Big(\int_0^\infty t|\nabla_\alpha P_t^\alpha f(x)|^2\, dt\Big)^{1/2}$$

and

$$\widetilde{g}_i(f)(x) = \Big(\int_0^\infty t|\partial_t \widetilde{P}_t^{\alpha,i} f(x)|^2\, dt\Big)^{1/2}, \qquad i = 1, \dots, d.$$

For a vector-valued function $F(x) = (f_1(x), \dots, f_d(x))$ we define

$$\widetilde{g}(F)(x) = \Big(\widetilde{g}_1(f_1)(x), \dots, \widetilde{g}_d(f_d)(x)\Big).$$

The main results of this section read as follows.

THEOREM 2.4.1. *Let $1 < p < \infty$, $\alpha \in (-1,\infty)^d$ and Λ be the function from Lemma 2.3.2. There exists a constant c_p such that*

$$\|g(f)\|_{L^p(d\mu_\alpha)} \le c_p \big[\Lambda(\min_i \alpha_i)\big]^2 \|f\|_{L^p(d\mu_\alpha)} \tag{2.9}$$

for all $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$.

THEOREM 2.4.2. *Let $1 < p < \infty$ and $\alpha \in [-1/2, \infty)^d$.*

(a) *Given $1 \le i \le d$ we have*

$$c_p^{-1}\|f\|_{L^p(d\mu_\alpha)} \le \|\widetilde{g}_i(f)\|_{L^p(d\mu_\alpha)} \le c_p \|f\|_{L^p(d\mu_\alpha)}$$

for all $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$.

(b) *Let $\mathcal{G} = \#\{\alpha_i : i = 1, \dots, d\}$ be the number of distinct coordinates in the multi-index α. Then*

$$\big\||F|_{\ell^2}\big\|_{L^p(d\mu_\alpha)} \le c_p \mathcal{G} \big\||\widetilde{g}(F)|_{\ell^2}\big\|_{L^p(d\mu_\alpha)}$$

for all $F(x) = (f_1(x), \dots, f_d(x))$ such that $|F|_{\ell^2} \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$.

2.4.1. Proof of Theorem 2.4.1.

For a reasonable function $F = F(t,x)$ define

$$\mathbb{L}^\alpha F(t,x) = \partial_t^2 F(t,x) - \mathcal{L}_x^\alpha F(t,x).$$

We will need several technical lemmas.

LEMMA 2.4.3. *Let $F = F(t,x)$ be a C^2 function mapping $\mathbb{R}_+ \times \mathbb{R}^d_+$ into $(0,\infty)$ such that $\mathbb{L}^\alpha F = 0$. Then for any $p \ge 1$ we have*

$$\mathbb{L}^\alpha(F^p) = p(p-1)F^{p-2}|\nabla_\alpha F|^2.$$

PROOF. The result follows by a simple computation, see [**GIT**]. □

LEMMA 2.4.4. *Let $F\colon \mathbb{R}_+ \times \mathbb{R}^d_+ \longrightarrow \mathbb{R}$ be a C^2 function such that $\mathbb{L}^\alpha F \ge 0$ or $\int_0^\infty \int_{\mathbb{R}^d_+} t\,|\mathbb{L}^\alpha F(t,x)|\, d\mu_\alpha(x) dt < \infty$. Assume that*

(a) $\sup\{|F(t,x)| : t > 0, x \in \mathbb{R}^d_+\} < \infty$;

(b) $\sup\{|\nabla_x F(t,x)| : t > 0, x \in \mathbb{R}^d_+\} < \infty$;

(c) *$t\,|\partial_t F(t,x)| \le c\,(1 + |x|)\varrho(t)$ for all $t > 0$, where the function ϱ is continuous, vanishes at 0 and ∞, and satisfies $\int_0^\infty t^{-1}\varrho(t)\, dt < \infty$.*

Then

$$\int_0^\infty \int_{\mathbb{R}^d_+} t\,\mathbb{L}^\alpha F(t,x)\, d\mu_\alpha(x)dt = \int_{\mathbb{R}^d_+} F(0,x)\, d\mu_\alpha(x) - \int_{\mathbb{R}^d_+} F(\infty,x)\, d\mu_\alpha(x).$$

PROOF. First observe that for each $x \in \mathbb{R}^d_+$ the limit

$$F(\infty,x) = \lim_{t\to\infty} F(t,x)$$

does exist. Indeed, by the condition (c) we may write

$$\begin{aligned}|F(t+T,x) - F(t,x)| &\le \sum_{j=0}^{N-1} \left| F\left(t + \frac{j+1}{N}T\right) - F\left(t+\frac{j}{N}T\right)\right| \\ &= \sum_{j=0}^{N-1} |\partial_t F(\theta_j,x)|\, \frac{T}{N} \le C\,(1+|x|) \sum_{j=0}^{N-1} \frac{T}{N}\frac{\varrho(\theta_j)}{\theta_j} = C\,(1+|x|)S_N,\end{aligned}$$

with $\theta_j \in \left(t+\frac{jT}{N}, t+\frac{(j+1)T}{N}\right)$. Since $S_N \longrightarrow \int_t^{t+T} r^{-1}\varrho(r)\, dr$ as $N\to\infty$, the conclusion follows. Similar arguments show the existence of the limit

$$F(0,x) = \lim_{t\to 0^+} F(t,x).$$

Now, observe that

$$\mathbb{L}^\alpha F(t,x) = \partial_t^2 F(t,x) + \sum_{j=1}^d e^{x_j} x_j^{-\alpha_j} \partial_{x_j}\left[e^{-x_j} x_j^{\alpha_j+1} \partial_{x_j} F(t,x)\right].$$

Let $\mathcal{D}_T = (e^{-T},T)\times(e^{-T},T)^d$. Given $j\in\{1,\dots,d\}$ we have

$$\begin{aligned}&\int_{e^{-T}}^T e^{x_j} x_j^{-\alpha_j}\partial_{x_j}\left[e^{-x_j}x_j^{\alpha_j+1}\partial_{x_j}F(t,x)\right]\, d\mu_{\alpha_j}(x_j) \\ &\qquad = e^{-T}T^{\alpha_j+1}\partial_{x_j}F(t,x)\Big|_{x_j=T} - \exp(-e^{-T})e^{-T(\alpha_j+1)}\partial_{x_j}F(t,x)\Big|_{x_j=e^{-T}},\end{aligned}$$

hence, by (b),

$$\left|\int_{\mathcal{D}_T} t e^{x_j} x_j^{-\alpha_j}\partial_{x_j}\left[e^{-x_j}x_j^{\alpha_j+1}\partial_{x_j}F(t,x)\right]\, d\mu_\alpha(x)dt\right| \le cT^2\left(e^{-T}T^{\alpha_j+1} + e^{-(\alpha_j+1)T}\right).$$

Therefore,

$$\int_{\mathcal{D}_T} t\sum_{j=1}^d e^{x_j}x_j^{-\alpha_j}\partial_{x_j}\left[e^{-x_j}x_j^{\alpha_j+1}\partial_{x_j}F(t,x)\right]\, d\mu_\alpha(x)dt \longrightarrow 0, \quad T\longrightarrow\infty.$$

This, together with the monotone (or the dominated) convergence theorem, implies

$$\int_0^\infty\int_{\mathbb{R}^d_+} t\,\mathbb{L}^\alpha F(t,x)\, d\mu_\alpha(x)dt = \lim_{T\to\infty}\int_{\mathcal{D}_T} t\partial_t^2 F(t,x)\, dtd\mu_\alpha(x).$$

On the other hand, integrating by parts we obtain

$$\int_{e^{-T}}^T t\partial_t^2 F(t,x)\, dt \;=\; F(e^{-T},x) - F(T,x) + t\partial_t F(t,x)\Big|_{t=e^{-T}}^{t=T}.$$

By (c) the absolute value of the last term is estimated by $c\,(1+|x|)[\varrho(e^{-T})+\varrho(T)]$. Since $\lim_{t\to 0^+}\varrho(t) = \lim_{t\to\infty}\varrho(t) = 0$ and $(1+|\cdot|)\in L^1(\mathbb{R}^d_+, d\mu_\alpha)$, the proof is finished with the aid of (a) and the dominated convergence theorem. □

PROPOSITION 2.4.5. *Lemma 2.4.4 may be applied to the function*

$$F(t,x) = (P_t^\alpha f(x))^p,$$

where $p \geq 1$ *and* f *is an arbitrary nonnegative function from* $C_c^2(\mathbb{R}_+^d)$.

PROOF. By the subordination principle and Proposition 2.3.4 we get

$$\begin{aligned} \partial_t P_t^\alpha f(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \partial_t \left[T_{t^2/(4u)}^\alpha f(x) \right] du \\ &= \frac{-1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} \mathcal{L}^\alpha \left[T_{t^2/(4u)}^\alpha f(x) \right] du. \end{aligned}$$

Interchanging the order of differentiation and integration above is justified by the dominated convergence theorem, using (see also the considerations below)

$$\begin{aligned} \partial_{x_j} T_t^\alpha f(x) &= e^{-t} T_t^{\alpha+e_j} \left(\partial_{x_j} f \right)(x), \\ \partial_{x_j}^2 T_t^\alpha f(x) &= e^{-2t} T_t^{\alpha+2e_j} \left(\partial_{x_j}^2 f \right)(x). \end{aligned}$$

These identities are easily verified for Laguerre polynomials, and for functions from $C_c^2(\mathbb{R}_+^d) \subset L^2(\mathbb{R}_+^d, d\mu_\alpha)$ they are checked by term by term differentiation of the series defining $T_t^\alpha f$, see the proof of Proposition 2.3.4. Thus

$$\begin{aligned} \partial_t P_t^\alpha(x) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(2u)} \sum_{j=1}^d x_j T_{t^2/(4u)}^{\alpha+2e_j} \left(\partial_{x_j}^2 f \right)(x)\, du \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(4u)} \sum_{j=1}^d (\alpha_j+1) T_{t^2/(4u)}^{\alpha+e_j} \left(\partial_{x_j} f \right)(x)\, du \\ &\quad - \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(4u)} \sum_{j=1}^d x_j T_{t^2/(4u)}^{\alpha+e_j} \left(\partial_{x_j} f \right)(x)\, du \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

Since T_t^α are contractions on $L^\infty(\mathbb{R}_+^d)$ and f has bounded first and second order derivatives we have

$$|t\mathcal{I}_1| \leq c\,t \sum_{j=1}^d x_j \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/(2u)}\, du = c\,t e^{-t\sqrt{2}} \sum_{j=1}^d x_j\,,$$

and similarly

$$|t\mathcal{I}_2| + |t\mathcal{I}_3| \leq c\,t e^{-t} \sum_{j=1}^d (\alpha_j+1) + c\,t e^{-t} \sum_{j=1}^d x_j.$$

Consequently,

$$|t\partial_t P_t^\alpha f(x)| \leq c(1+|x|) t e^{-t}.$$

Since f is bounded and P_t^α are contractions on $L^\infty(\mathbb{R}_+^d)$ (which follows by the same property for T_t^α and the subordination principle), the hypotheses (a) and (c) of Lemma 2.4.4 are satisfied (with $\varrho(t) = te^{-t}$). Concerning (b), we have

$$\begin{aligned} \left|\partial_{x_j} P_t^\alpha f(x)\right| &= \left| \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/(4u)} T_{t^2/(4u)}^{\alpha+e_j} \left(\partial_{x_j} f \right)(x)\, du \right| \\ &\leq \frac{c}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/(4u)}\, du \;=\; c\,e^{-t}. \end{aligned}$$

The fact that $\mathbb{L}^\alpha\big[(P_t^\alpha f(x))^p\big] \geq 0$ is justified with the aid of Proposition 2.3.4 and Lemma 2.4.3 (an application of Lemma 2.4.3 is possible, because $P_t^\alpha f$ is strictly positive if $f(x_0) > 0$ for some $x_0 \in \mathbb{R}_+^d$, see the comment in the proof of Theorem 2.4.1 below). □

PROPOSITION 2.4.6. *Let $f \in C_c^2(\mathbb{R}_+^d)$ be nonnegative. There exists a constant c (depending on f) such that*

$$g(f)(x) \leq c(1+|x|), \qquad x \in \mathbb{R}_+^d,$$

hence $g(f) \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$ for all $1 \leq p < \infty$.

PROOF. We apply the estimates on $|\partial_t P_t^\alpha f(x)|$ and $|\partial_{x_j} P_t^\alpha f(x)|$ obtained in the proof of Proposition 2.4.5. □

PROPOSITION 2.4.7. *Lemma 2.4.4 may be applied to the function*

$$F(t,x) = (P_t^\alpha f(x))^2 P_t^\alpha h(x),$$

with arbitrary nonnegative $f, h \in C_c^2(\mathbb{R}_+^d)$.

PROOF. Items (a)-(c) are verified similarly as in the proof of Proposition 2.4.5. It remains to show the integrability condition.

Given reasonable functions $F, G\colon \mathbb{R}_+ \times \mathbb{R}_+^d \longrightarrow \mathbb{R}$, one has

$$\mathbb{L}^\alpha(FG) = (\mathbb{L}^\alpha F)G + (\mathbb{L}^\alpha G)F + 2\langle \nabla_\alpha F, \nabla_\alpha G\rangle. \tag{2.10}$$

Therefore,

$$\left|\mathbb{L}^\alpha[(P_t^\alpha f(x))^2 P_t^\alpha h(x)]\right| \leq \mathbb{L}^\alpha(P_t^\alpha f(x))^2\, P_t^\alpha h(x) + 2\left|\nabla_\alpha (P_t^\alpha f(x))^2\right| \left|\nabla_\alpha P_t^\alpha h(x)\right|,$$

since $\mathbb{L}^\alpha(P_t^\alpha h(x)) = 0$ by Proposition 2.3.4. Now, observe that

$$\int_0^\infty \int_{\mathbb{R}_+^d} t\, \mathbb{L}^\alpha(P_t^\alpha f(x))^2\, P_t^\alpha h(x)\, d\mu_\alpha(x) dt \leq c \int_0^\infty \int_{\mathbb{R}_+^d} t\, \mathbb{L}^\alpha(P_t^\alpha f(x))^2 d\mu_\alpha(x) dt$$

and the last term is finite by Proposition 2.4.5. Further, by Schwarz' inequality and the fact that $P_t^\alpha f(x)$ is bounded we obtain

$$\int_0^\infty \int_{\mathbb{R}_+^d} t\left|\nabla_\alpha (P_t^\alpha f(x))^2\right| \left|\nabla_\alpha P_t^\alpha h(x)\right| d\mu_\alpha(x) dt \leq c \int_{\mathbb{R}_+^d} g(f)(x)\, g(h)(x)\, d\mu_\alpha(x),$$

the last integral being finite by Proposition 2.4.6. □

PROOF OF THEOREM 2.4.1; THE CASE $1 < p \leq 2$.
Apart from minor changes, the proof relies on a classical reasoning, see [**S1**]. Observe first, that it is sufficient to prove (2.9) for all nonnegative $f \in C_c^2(\mathbb{R}_+^d)$. Then the theorem is justified by standard arguments, that is decomposition into positive and negative parts, approximation of each part by an increasing sequence of nonnegative smooth compactly supported functions, and an application of Fatou's lemma.

Assume that $f \in C_c^2(\mathbb{R}_+^d)$, $f \geq 0$ and $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}_+^d$. Since the kernel $G_t^\alpha(x,y)$ is strictly positive we have $T_t^\alpha f(x) > 0$, $x \in \mathbb{R}_+^d$ and hence, by the subordination formula, $P_t^\alpha f(x) > 0$, $x \in \mathbb{R}_+^d$. An application of Lemma 2.4.3 gives

$$[g(f)(x)]^2 \;=\; \int_0^\infty t|\nabla_\alpha (P_t^\alpha f(x))|^2\, dt$$

$$\begin{aligned} &= \frac{1}{p(p-1)} \int_0^\infty t (P_t^\alpha f(x))^{2-p}\, \mathbb{L}^\alpha (P_t^\alpha f(x))^p \, dt \\ &\leq \frac{1}{p(p-1)} [(P^\alpha)^* f(x)]^{2-p} \int_0^\infty t\, \mathbb{L}^\alpha (P_t^\alpha f(x))^p \, dt. \end{aligned}$$

Thus, by Hölder's inequality, (2.2) and Proposition 2.4.5 we obtain

$$\begin{aligned} \|g(f)\|^p_{L^p(d\mu_\alpha)} &\leq c_p \int_{\mathbb{R}^d_+} [(P^\alpha)^* f(x)]^{p(1-p/2)} \left(\int_0^\infty t\, \mathbb{L}^\alpha (P_t^\alpha f(x))^p \, dt \right)^{p/2} d\mu_\alpha(x) \\ &\leq c_p \|f\|^{p(1-p/2)}_{L^p(d\mu_\alpha)} \left(\int_0^\infty \int_{\mathbb{R}^d_+} t\, \mathbb{L}^\alpha (P_t^\alpha f(x))^p \, d\mu_\alpha(x) dt \right)^{p/2} \\ &= c_p \|f\|^{p(1-p/2)}_{L^p(d\mu_\alpha)} \left[\int_{\mathbb{R}^d_+} f(x)^p \, d\mu_\alpha(x) - \int_{\mathbb{R}^d_+} (P_\infty^\alpha f(x))^p \, d\mu_\alpha(x) \right]^{p/2} \\ &\leq c_p \|f\|^p_{L^p(d\mu_\alpha)}, \end{aligned}$$

since $\mathbb{L}^\alpha (P_t^\alpha f(x))^p \geq 0$, see the proof of Proposition 2.4.5. The conclusion follows. □

Proof of Theorem 2.4.1; the case $2 < p < \infty$.
We begin with some basic observations. Given a reasonable function f, by Schwarz' inequality we have

$$\left(P_t^\alpha f(x)\right)^2 \leq P_t^\alpha (f^2)(x) P_t^\alpha \mathbf{1}(x) = P_t^\alpha (f^2)(x).$$

Similarly, by Corollary 2.3.3 we get

$$\left(\widetilde{P}_t^{\alpha,i} f(x)\right)^2 \leq \widetilde{P}_t^{\alpha,i} (f^2)(x) \widetilde{P}_t^{\alpha,i} \mathbf{1}(x) \leq \left[\Lambda(\alpha_i)\right]^2 P_t^\alpha (f^2)(x).$$

Assume that $f \in C_c^1(\mathbb{R}^d_+)$. Then

$$\partial_t P_{t+s}^\alpha f(x) = P_s^\alpha \left(\partial_t P_t^\alpha f\right)(x).$$

Moreover,

$$\delta_j (P_t^\alpha f)(x) = \widetilde{P}_t^{\alpha,j} (\delta_j f)(x), \qquad j = 1, \dots, d,$$

which is directly verified for Laguerre polynomials, and for $f \in C_c^1(\mathbb{R}^d_+)$ follows by differentiating term by term the series of $P_t^\alpha f$. Therefore

$$\delta_j (P_{t+s}^\alpha f)(x) = \widetilde{P}_t^{\alpha,j} (\delta_j P_s^\alpha f)(x), \qquad j = 1, \dots, d.$$

Let f, h be nonnegative functions from $C_c^2(\mathbb{R}^d_+)$ and denote $\mathcal{C}(\alpha) = \left[\Lambda(\min_i \alpha_i)\right]^2$. Using the above observations, symmetry of P_t^α and Lemma 2.4.3 with $p = 2$ we write

$$\begin{aligned} &\int_{\mathbb{R}^d_+} [g(f)(x)]^2 h(x)\, d\mu_\alpha(x) = \int_{\mathbb{R}^d_+} \int_0^\infty t |\nabla_\alpha P_t^\alpha f(x)|^2 h(x) \, dt d\mu_\alpha(x) \\ &= \int_{\mathbb{R}^d_+} \int_0^\infty t \left[\left(P_{t/2}^\alpha \left(\partial_t P_{t/2}^\alpha f \right)(x) \right)^2 + \sum_{j=1}^d \left(\widetilde{P}_{t/2}^{\alpha,j} \left(\delta_j P_{t/2}^\alpha f \right)(x) \right)^2 \right] h(x)\, dt d\mu_\alpha(x) \\ &\leq \mathcal{C}(\alpha) \int_{\mathbb{R}^d_+} \int_0^\infty t \left[P_{t/2}^\alpha \left(\left[\partial_t P_{t/2}^\alpha f \right]^2 \right)(x) + \sum_{j=1}^d P_{t/2}^\alpha \left(\left[\delta_j P_{t/2}^\alpha f \right]^2 \right)(x) \right] h(x)\, dt d\mu_\alpha(x) \\ &= \mathcal{C}(\alpha) \int_0^\infty t \int_{\mathbb{R}^d_+} \left| \nabla_\alpha P_{t/2}^\alpha f(x) \right|^2 P_{t/2}^\alpha h(x)\, d\mu_\alpha(x) dt \end{aligned}$$

$$= 2\mathcal{C}(\alpha) \int_0^\infty t \int_{\mathbb{R}^d_+} \mathbb{L}^\alpha \left[P_t^\alpha f(x)\right]^2 P_t^\alpha h(x)\, d\mu_\alpha(x) dt.$$

Now, by (2.10) and the identity $\mathbb{L}^\alpha(P_t^\alpha h(x)) = 0$, the last double integral equals

$$\int_0^\infty t \int_{\mathbb{R}^d_+} \mathbb{L}^\alpha \Big[(P_t^\alpha f(x))^2\, P_t^\alpha h(x) \Big]\, d\mu_\alpha(x) dt$$

$$- \; 2 \int_0^\infty t \int_{\mathbb{R}^d_+} \langle \nabla_\alpha P_t^\alpha h(x), \nabla_\alpha (P_t^\alpha f(x))^2 \rangle \, d\mu_\alpha(x) dt \; = \; \mathcal{J}_1 - \mathcal{J}_2.$$

In view of the estimates from the proof of Proposition 2.4.7 the integrals $\mathcal{J}_1$ and $\mathcal{J}_2$ are absolutely convergent. By Proposition 2.4.7

$$\mathcal{J}_1 \le \int_{\mathbb{R}^d_+} f(x)^2 h(x)\, d\mu_\alpha(x)$$

and, by Schwarz' inequality,

$$|\mathcal{J}_2| \le 4 \int_{\mathbb{R}^d_+} [(P^\alpha)^* f](x)\; g(f)(x)\; g(h)(x)\; d\mu_\alpha(x).$$

Thus,

$$\int_{\mathbb{R}^d_+} [g(f)(x)]^2 h(x)\, d\mu_\alpha(x) \le$$

$$2\mathcal{C}(\alpha) \int_{\mathbb{R}^d_+} f^2(x) h(x)\, d\mu_\alpha(x) + 8\mathcal{C}(\alpha) \int_{\mathbb{R}^d_+} [(P^\alpha)^* f](x)\; g(f)(x)\; g(h)(x)\; d\mu_\alpha(x).$$

Assume that $p \ge 4$, $(2/p) + (1/q) = 1$ and $\|h\|_{L^q(d\mu_\alpha)} \le 1$. By Hölder's inequality (for three functions), (2.2) and Theorem 2.4.1 for $q \le 2$, we obtain

$$\langle g(f)^2, h \rangle_{d\mu_\alpha} \le c_p\, \mathcal{C}(\alpha) \left(\|f\|^2_{L^p(d\mu_\alpha)} + \|g(f)\|_{L^p(d\mu_\alpha)} \|f\|_{L^p(d\mu_\alpha)} \right),$$

hence

$$\|g(f)\|^2_{L^p(d\mu_\alpha)} \le c_p\, \mathcal{C}(\alpha) \left(\|f\|^2_{L^p(d\mu_\alpha)} + \|g(f)\|_{L^p(d\mu_\alpha)} \|f\|_{L^p(d\mu_\alpha)} \right).$$

This implies the desired estimate for $p \ge 4$. For $2 < p < 4$ the result follows by Marcinkiewicz' interpolation theorem. □

2.4.2. Proof of Theorem 2.4.2.

To prove the upper bound in Theorem 2.4.2 (a) one could try to adopt the reasoning used in the proof of Theorem 2.4.1, as was done by Gutiérrez [**Gu**] in the Hermite setting. Indeed, such an adaptation is (at least partially) possible, but the corresponding analysis is considerably more involved. Therefore we prefer to invoke a general result, which immediately implies the desired inequalities.

Since for $\alpha \in [-1/2, \infty)^d$ the semigroups $\{\widetilde{P}_t^{\alpha,i}\}$, $i = 1, \dots, d$, form positive symmetric contraction semigroups (see Corollary 2.3.3), Theorem 2.4.2 (a) is a consequence of the refinement of Stein's general Littlewood-Paley theory [**S1**], which is due to Coifman, Rochberg and Weiss [**CRW**], see also Meda [**M**, Theorem 2].

The case when $\alpha \notin [-1/2, \infty)^d$ seems to require more subtle treatment. Here are some details. Fix $i \in \{1, \dots, d\}$. Notice that the function $x^{-1/2}$ belongs to

$L^2(\mathbb{R}_+, d\mu_{\alpha_i+1})$ for all $\alpha_i \in (-1, \infty)$. Hence it has the Fourier-Laguerre expansion

$$x^{-1/2} = \sum_{k\in\mathbb{N}} a_k^i L_k^{\alpha_i+1}.$$

The coefficients a_k^i are computed to be (see [**Le**, Sec.4.24])

$$a_k^i = \frac{\Gamma(\alpha_i + 3/2)}{\Gamma(\alpha_i + 2)} \frac{(1/2)_k}{(\alpha_i + 2)_k},$$

where $(\cdot)_k$ is the Pochhammer symbol. Therefore $\mathbf{1}$ has the expansion

$$\mathbf{1} = \frac{\Gamma(\alpha_i + 3/2)}{\Gamma(\alpha_i + 2)} \sum_{k\in\mathbb{N}} \frac{(1/2)_k}{(\alpha_i + 2)_k} \sqrt{x_i} L_k^{\alpha_i+1},$$

the series being convergent in $L^2(\mathbb{R}_+, d\mu_{\alpha_i})$ since $(a_k^i) \in \ell^2$. It follows that

$$\widetilde{T}_t^{\alpha,i}\mathbf{1} = \frac{\Gamma(\alpha_i + 3/2)}{\Gamma(\alpha_i + 2)} \sum_{k\in\mathbb{N}} \frac{(1/2)_k}{(\alpha_i + 2)_k} e^{-t(k+1)} \sqrt{x_i} L_k^{\alpha_i+1}.$$

Now, using suitable generating formula (see [**SrMa**, Ch.2, Sec.2.5]) we obtain

$$\widetilde{T}_t^{\alpha,i}\mathbf{1}(x) = \frac{\Gamma(\alpha_i + 3/2)}{\Gamma(\alpha_i + 2)} e^{-t/2} \left(\frac{x_i e^{-t}}{1 - e^{-t}}\right)^{1/2} {}_1F_1\left(\frac{1}{2};\ \alpha_i + 2;\ -\frac{x_i e^{-t}}{1 - e^{-t}}\right),$$

where ${}_1F_1$ denotes the confluent hypergeometric function. It can be shown that

$$\frac{\Gamma(\alpha_i + 3/2)}{\Gamma(\alpha_i + 2)} \sup_{y>0} \sqrt{y}\, {}_1F_1\big(1/2;\ \alpha_i + 2;\ -y\big)$$

does not exceed 1 if $\alpha_i \geq -1/2$ and is greater than 1 if $\alpha_i < -1/2$. Thus $\widetilde{T}_t^{\alpha,i}$ are not contractions on $L^\infty(\mathbb{R}_+^d)$ for t sufficiently small. Since $\|\cdot\|_{L^p(d\mu_\alpha)} \longrightarrow \|\cdot\|_\infty$ as $p \longrightarrow \infty$, we see that $\widetilde{T}_t^{\alpha,i}$ are not contractions on $L^p(\mathbb{R}_+^d, d\mu_\alpha)$ for t sufficiently small and p sufficiently large.

The above example shows, that $\alpha_i = -1/2$ is a "critical" point for the contraction properties of $\{\widetilde{T}_t^{\alpha,i}\}$ (noteworthy, the same critical point appears in the logarithmic Sobolev inequality related to the Laguerre semigroup, see [**Ko**]). Nevertheless, it may happen that contraction properties of $\{\widetilde{P}_t^{\alpha,i}\}$ are preserved for $\alpha_i < -1/2$. This, however, is by no means obvious and seems to require a distinct detailed analysis. Let us also mention, that we were not able to treat the case $\alpha_i < -1/2$ by suitable adaptation of the proof of Theorem 2.4.2. The reason for that is, roughly speaking, that the difference

$$M_i^\alpha - \mathcal{L}^\alpha = \frac{\alpha_i + 1/2 + x_i}{2x_i}$$

is "negative" for small x_i if $\alpha_i < -1/2$.

The proof of Theorem 2.4.2 (b) is a straightforward modification of the arguments given in [**Gu**]. We present it with details for the sake of completeness. To the end of this subsection we assume that $\alpha \in [-1/2, \infty)^d$.

Fix $i \in \{1, \dots, d\}$. If $f \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$ has the expansion $f = \sum_k a_k^i(f) \sqrt{x_i} L_k^{\alpha+e_i}$, then

$$\widetilde{P}_t^{\alpha,i} f = \sum_{k\in\mathbb{N}^d} a_k^i(f) e^{-t(|k|+1)^{1/2}} \sqrt{x_i} L_k^{\alpha+e_i}$$

and therefore

$$\partial_t \widetilde{P}_t^{\alpha,i} f = - \sum_{k\in\mathbb{N}^d} a_k^i(f)(|k|+1)^{1/2} e^{-t(|k|+1)^{1/2}} \sqrt{x_i} L_k^{\alpha+e_i}.$$

Using Parseval's identity we get

$$\int_{\mathbb{R}_+^d} \left(\partial_t \widetilde{P}_t^{\alpha,i} f(x)\right)^2 d\mu_\alpha(x) = \sum_{k\in\mathbb{N}^d} a_k^i(f)^2(|k|+1)e^{-2t(|k|+1)^{1/2}} \|\sqrt{y_i}L_k^{\alpha+e_i}\|^2_{L^2(d\mu_\alpha)}.$$

Hence, applying again Parseval's identity,

$$\begin{aligned}\int_{\mathbb{R}_+^d} [\widetilde{g}_i(f)(x)]^2 \, d\mu_\alpha(x) &= \int_0^\infty t \int_{\mathbb{R}_+^d} \left[\partial_t \widetilde{P}_t^{\alpha,i} f(x)\right]^2 d\mu_\alpha(x)dt \\ &= \sum_{k\in\mathbb{N}^d} a_k^i(f)^2(|k|+1)\|\sqrt{y_i}L_k^{\alpha+e_i}\|^2_{L^2(d\mu_\alpha)} \int_0^\infty te^{-2t\sqrt{|k|+1}}dt \\ &= \frac{1}{4}\sum_{k\in\mathbb{N}^d} a_k^i(f)^2 \|\sqrt{y_i}L_k^{\alpha+e_i}\|^2_{L^2(d\mu_\alpha)} \;=\; \frac{1}{4}\|f\|^2_{L^2(d\mu_\alpha)}.\end{aligned}$$

This gives $2\|\widetilde{g}_i(f)\|_{L^2(d\mu_\alpha)} = \|f\|_{L^2(d\mu_\alpha)}$ and by polarization we obtain

$$4\int_0^\infty \int_{\mathbb{R}_+^d} t\left[\partial_t \widetilde{P}_t^{\alpha,i} f_1(x)\right]\left[\partial_t \widetilde{P}_t^{\alpha,i} f_2(x)\right] \, d\mu_\alpha(x)dt = \langle f_1, f_2\rangle_{d\mu_\alpha}, \tag{2.11}$$

for all real-valued $f_1, f_2 \in L^2(\mathbb{R}_+^d, d\mu_\alpha)$.

Let $F(x) = (f_1(x), \dots, f_d(x))$ be a vector-valued function in $\mathbb{R}_+^d$. For $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $|\xi| = 1$, we define

$$F_\xi(x) = F \circ \xi = \sum_{j=1}^d \xi_j f_j(x).$$

Assume that $1 < p < \infty$ and $1/p + 1/p' = 1$. Let $h \in L^2(\mathbb{R}_+^d, d\mu_\alpha) \cap L^{p'}(\mathbb{R}_+^d, d\mu_\alpha)$ be such that $\|h\|_{L^{p'}(d\mu_\alpha)} \le 1$. By (2.11) we get

$$\begin{aligned}\int_{\mathbb{R}_+^d} F_\xi(x)h(x) \, d\mu_\alpha(x) &= 4\int_0^\infty \int_{\mathbb{R}_+^d} t\sum_{i=1}^d \xi_i \left[\partial_t \widetilde{P}_t^{\alpha,i} f_i(x)\right]\left[\partial_t \widetilde{P}_t^{\alpha,i} h(x)\right] \, d\mu_\alpha(x)dt \\ &= 4\int_0^\infty \int_{\mathbb{R}_+^d} t\big|V(t,x)\big|_{\ell^2} \cos(\xi, V(t,x)) \, d\mu_\alpha(x)dt,\end{aligned}$$

where

$$V(t,x) = \left[\partial_t \widetilde{P}_t^{\alpha,i} f_i(x) \; \partial_t \widetilde{P}_t^{\alpha,i} h(x)\right]_{i=1}^d.$$

Denote by σ the surface measure on the unit sphere $S^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. Using the above and Minkowski's integral inequality we obtain

$$\begin{aligned}\big\|\langle F_\cdot, h\rangle_{d\mu_\alpha}\big\|_{L^p(S^{d-1},d\sigma)} &\le 4\int_{\mathbb{R}_+^d}\int_0^\infty t|V(t,x)|\big\|\cos(\cdot, V(t,x))\big\|_{L^p(S^{d-1},d\sigma)} dtd\mu_\alpha(x) \\ &= 4C(d,p)\int_{\mathbb{R}_+^d}\int_0^\infty t|V(t,x)| \; dtd\mu_\alpha(x).\end{aligned}$$

Let $\alpha_{i_1}, \dots, \alpha_{i_n}$ be all distinct values of coordinates in the multi-index α. Observe, that the last expression may be estimated from above by

$$4C(d,p)\sum_{j=1}^{n}\int_{\mathbb{R}^d_+}\int_0^\infty t\left|\partial_t \widetilde{P}_t^{\alpha,i_j}h(x)\right|\left(\sum_{i=1}^{d}\left[\partial_t\widetilde{P}_t^{\alpha,i}f_i(x)\right]^2\right)^{1/2} dt d\mu_\alpha(x),$$

which, by Hölder's inequality and Theorem 2.4.2 (a), is further bounded by

$$\begin{aligned} & 4C(d,p)\sum_{j=1}^{n}\int_{\mathbb{R}^d_+}\widetilde{g}_{i_j}(h)(x)|\widetilde{g}(F)(x)|\, d\mu_\alpha(x) \\ \leq\ & 4C(d,p)\sum_{j=1}^{n}\left\|\widetilde{g}_{i_j}(h)\right\|_{L^{p'}(d\mu_\alpha)}\left\||\widetilde{g}(F)|_{\ell^2}\right\|_{L^p(d\mu_\alpha)} \\ \leq\ & 4c_pC(d,p)n\left\||\widetilde{g}(F)|_{\ell^2}\right\|_{L^p(d\mu_\alpha)}. \end{aligned}$$

For each $\xi \in \mathbb{R}^d$, $|\xi| = 1$, there exists a sequence $\{h_m^\xi\} \subset L^2(\mathbb{R}^d_+, d\mu_\alpha) \cap L^{p'}(\mathbb{R}^d_+, d\mu_\alpha)$, $\|h_m^\xi\|_{L^{p'}(d\mu_\alpha)} \leq 1$, such that

$$\lim_m \left|\langle F_\xi, h_m^\xi\rangle_{d\mu_\alpha}\right| = \left\|F_\xi\right\|_{L^p(d\mu_\alpha)}.$$

Thus, by Fatou's lemma, we get

$$\int_{|\xi|=1}\|F_\xi\|^p_{L^p(d\mu_\alpha)}\, d\sigma(\xi) \leq \left(4c_pC(d,p)n\left\||\widetilde{g}(F)|_{\ell^2}\right\|_{L^p(d\mu_\alpha)}\right)^p. \tag{2.12}$$

Now, observe that

$$|F_\xi(x)|^p = |F(x)|^p_{\ell^2}\left|\cos(\xi, V(t,x))\right|^p,$$

and therefore

$$\int_{|\xi|=1}\|F_\xi\|^p_{L^p(d\mu_\alpha)}\, d\sigma(\xi) = C(d,p)^p\left\||F|_{\ell^2}\right\|^p_{L^p(d\mu_\alpha)}.$$

This, in view of (2.12), completes the proof.

2.5. Riesz-Laguerre transforms and conjugate Poisson integrals

Recall that the Riesz-Laguerre transform $\mathcal{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ is formally given by

$$\mathcal{R}^\alpha = \operatorname{grad}_\alpha(\mathcal{L}^\alpha)^{-1/2}\Pi_0,$$

which makes sense for a dense subset of $L^2(\mathbb{R}^d_+, d\mu_\alpha)$ of all polynomials, because for Laguerre polynomials we have

$$R_i^\alpha L_k^\alpha = -|k|^{-1/2}\sqrt{x_i}L_{k-e_i}^{\alpha+e_i}, \qquad |k| > 0,$$

and $R_i^\alpha L_k^\alpha = 0$ if $|k| = 0$. Given a suitable function f we define its *conjugate Poisson integrals* $U_t^{\alpha,i}f$, $i = 1, \dots, d$, $t > 0$, by

$$U_t^{\alpha,i}f = \widetilde{P}_t^{\alpha,i}R_i^\alpha f,$$

Such definition extends that given by Muckenhoupt [**Mu3**] in the one-dimensional setting and is well motivated by the following set of Cauchy-Riemann type equations:

$$\delta_j U_t^{\alpha,i}f = \delta_i U_t^{\alpha,j}f, \qquad i,j = 1, \dots, d, \tag{2.13}$$

$$\delta_j P_t^\alpha f = -\partial_t U_t^{\alpha,j}f, \qquad j = 1, \dots, d, \tag{2.14}$$

$$\sum_{j=1}^{d} \delta_j^* U_t^{\alpha,j} f \;=\; -\partial_t P_t^\alpha f. \tag{2.15}$$

Moreover,

$$\partial_t^2 U_t^{\alpha,j} f = M_j^\alpha U_t^{\alpha,j} f, \qquad j = 1, \dots, d. \tag{2.16}$$

Some of the above equations, in one-dimensional version, may be found in [**Mu3**]. Verification of (2.13)-(2.16) when f is a (Laguerre) polynomial is straightforward. Indeed, for $i = 1, \dots, d$, we have (see Sections 2.2 and 2.3)

$$\begin{aligned} P_t^\alpha L_k^\alpha &= e^{-t|k|^{1/2}}, \\ M_i^\alpha(\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) &= |k| \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \\ \widetilde{P}_t^{\alpha,i}(\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) &= e^{-t|k|^{1/2}} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \\ U_t^{\alpha,i} L_k^\alpha &= -|k|^{-1/2} e^{-t|k|^{1/2}} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i}, \qquad |k| > 0, \end{aligned}$$

and $U_t^{\alpha,i} L_k^\alpha = 0$ if $|k| = 0$. Now, identities (2.13) and (2.14) are easily verified with the aid of (2.4). To get (2.15) observe that $\delta_i^*(\sqrt{x_i} L_{k-e_i}^{\alpha+e_i}) = -k_i L_k^\alpha$, which follows by (2.4) and the fact that Laguerre polynomials L_k^α are eigenfunctions of $\mathcal{L}^\alpha$. Checking (2.16) makes no difficulties.

The main result of this chapter is the following.

THEOREM 2.5.1. *Assume that $1 < p < \infty$ and $\alpha \in [-1/2, \infty)$. Let $\mathcal{G} = \#\{\alpha_i : i = 1, \dots, d\}$ be the number of distinct coordinates in the multi-index α. There exists a constant c_p (depending neither on the dimension d nor on the type multi-index α) such that*

$$\big\| |\mathcal{R}^\alpha f|_{\ell^2} \big\|_{L^p(d\mu_\alpha)} \le c_p \mathcal{G} \|f\|_{L^p(d\mu_\alpha)}$$

and

$$\|R_i^\alpha f\|_{L^p(d\mu_\alpha)} \le c_p \|f\|_{L^p(d\mu_\alpha)}, \qquad i = 1, \dots, d,$$

for all polynomials f in $\mathbb{R}_+^d$.

PROOF. In view of (2.14) we have

$$\big[\partial_t \widetilde{P}_t^{\alpha,1}(R_1^\alpha f), \dots, \partial_t \widetilde{P}_t^{\alpha,d}(R_d^\alpha f)\big] = \big[-\delta_1 P_t^\alpha f, \dots, -\delta_d P_t^\alpha f\big],$$

hence

$$\big|\widetilde{g}(\mathcal{R}^\alpha f)(x)\big|_{\ell^2} = \big|[\widetilde{g}_1(R_1^\alpha f)(x), \dots, \widetilde{g}_d(R_d^\alpha f)(x)]\big|_{\ell^2} \le g(f)(x).$$

Ergo the conclusion follows by Theorems 2.4.1 and 2.4.2. □

COROLLARY 2.5.2. *Let $\alpha \in [-1/2, \infty)$. The operators R_i^α and $U_t^{\alpha,i}$, $i = 1, \dots, d$, $t > 0$, initially defined on a dense subset of $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, extend uniquely to bounded linear operators in $L^p(\mathbb{R}_+^d, d\mu_\alpha)$, $1 < p < \infty$.*

COROLLARY 2.5.3. *Let $1 < p < \infty$ and $\alpha \in [-1/2, \infty)$. Then*

(a) *There exists a constant c_p (independent of d and α) such that*

$$\|U_t^{\alpha,i} f\|_{L^p(d\mu_\alpha)} \le c_p \|f\|_{L^p(d\mu_\alpha)}, \qquad i = 1, \dots, d,$$

for all $f \in L^p(\mathbb{R}_+^d, d\mu_\alpha)$;

(b) *For all* $f \in L^p(\mathbb{R}^d_+, d\mu_\alpha)$ *and* $i \in \{1, \dots, d\}$

$$U_t^{\alpha,i} f \longrightarrow R_i^\alpha f, \quad t \longrightarrow 0^+,$$

the convergence being both in $L^p(\mathbb{R}^d_+, d\mu_\alpha)$ *and almost everywhere;*

(c) *For each* $i \in \{1, \dots, d\}$ *the family* $\{U_t^{\alpha,i}\}_{t>0}$ *is strongly continuous and uniformly bounded in* $L^p(\mathbb{R}^d_+, d\mu_\alpha)$.

PROOF. Item (a) is a consequence of Theorem 2.5.1, (2.2) and the fact that

$$\sup_{t>0} |U_t^{\alpha,i} f(x)| \le (P^\alpha)^* \big(R_i^\alpha f\big)(x), \qquad x \in \mathbb{R}^d_+.$$

Statements (b) and (c) are justified by standard arguments with the aid of (a). □

REMARK 2.5.4. When $p = 2$ we have

$$\big\| |\mathcal{R}^\alpha f|_{\ell^2} \big\|_{L^2(d\mu_\alpha)} = \big\| \Pi_0 f \big\|_{L^2(d\mu_\alpha)}, \qquad f \in L^2(\mathbb{R}^d_+, d\mu_\alpha), \quad \alpha \in (-1, \infty)^d.$$

Indeed, if $f \in L^2(\mathbb{R}^d_+, d\mu_\alpha)$ has the expansion $f = \sum_k a_k(f) L_k^\alpha$, then

$$R_i^\alpha f = - \sum_{|k|>0} a_k(f) |k|^{-1/2} \sqrt{x_i} L_{k-e_i}^{\alpha+e_i},$$

and therefore by Parseval's identity

$$\begin{aligned} \big\| |\mathcal{R}^\alpha f|_{\ell^2} \big\|^2_{L^2(d\mu_\alpha)} &= \sum_{i=1}^d \sum_{|k|>0} a_k(f)^2 |k|^{-1} \| \sqrt{y_i} L_{k-e_i}^{\alpha+e_i} \|^2_{L^2(d\mu_\alpha)} \\ &= \sum_{i=1}^d \sum_{|k|>0} a_k(f)^2 k_i |k|^{-1} \| L_k^\alpha \|^2_{L^2(d\mu_\alpha)} \\ &= \sum_{|k|>0} a_k(f)^2 \| L_k^\alpha \|^2_{L^2(d\mu_\alpha)} = \| \Pi_0 f \|^2_{L^2(d\mu_\alpha)}. \end{aligned}$$

Analogous computations show that for each $i \in \{1, \dots, d\}$ the operators $\{U_t^{\alpha,i}\}_{t>0}$ are contractions in $L^2(\mathbb{R}^d_+, d\mu_\alpha)$, $\alpha \in (-1, \infty)^d$.

CHAPTER 3

Higher order Riesz transforms for polynomial Laguerre expansions

3.1. Introduction

In the present chapter we focus on higher order Riesz transforms and some related operators for multi-dimensional polynomial Laguerre expansions. Such operators do not appear in the existing literature, thus we start by giving suitable definitions. The main results, boundedness in L^p, $1 < p < \infty$, of higher order Riesz-Laguerre transforms (Theorem 3.5.1) and weak type 1–1 of Riesz-Laguerre transforms of order 2 (Theorem 3.5.2), are obtained by means of transference from Hermite polynomial setting, after restricting to half-integer multi-indices α. The corresponding L^p constants depend neither on the dimension nor on the type multi-index α.

The method of transference was used by Dinger [**Di**] and other authors, and recently has been developed by Gutiérrez et al., [**GIT**]. We provide a considerable extension of this technique and show how to transfer higher order Riesz type operators and certain differential operators. Although the corresponding formulas are rather complex, we believe they shed some new light on an interplay between Hermite and Laguerre expansions. The crucial ingredient of our reasoning is technical Lemma 3.4.3. This result is of independent interest and may be applied to derive certain bilinear composition formulas for multi-dimensional Gould-Hopper polynomials, hence also for multi-dimensional Hermite polynomials (see [**GrNo**]). Other interesting applications of the results obtained in Chapter 3 have been recently given in [**GLLNU**], in investigation of Sobolev spaces associated with polynomial Laguerre expansions.

The organization of this chapter is the following. In Section 3.2 we furnish relevant definitions and describe the transference setting. Section 3.3 contains two examples which are given to provide a better insight in what follows later. These are explicit computations of transference inequality for Riesz-Laguerre transforms of order 2 and 3. In Section 3.4 we obtain a transference inequality for Riesz-Laguerre transforms and related operators of arbitrary finite order . Finally, in Section 3.5 we state and prove main results concerning L^p boundedness and weak type 1–1 of the aforementioned operators.

3.2. Definitions and transference setting

We shall use the notation of Chapter 2. Recall that α is the type multi-index, $\mathcal{L}^\alpha$ is the Laguerre differential operator, μ_α is the associated measure in $\mathbb{R}^d_+$, and Π_0 denotes the orthogonal projection onto the orthogonal complement $\mathcal{H}_0^\perp$ of the eigenspace corresponding to the Laguerre eigenvalue 0. Further, δ_i, $i = 1, \dots, d$ are the Laguerre partial derivatives defined by $\delta_i = \sqrt{x_i}\partial_{x_i}$.

Let $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ be a multi-index. Using the standard notation $\delta^m = \delta_1^{m_1} \cdots \delta_d^{m_d}$ it is natural to define the *Riesz-Laguerre transform of order* $M \in \mathbb{N}$

by

$$\mathcal{R}^{\alpha,M} = (R^\alpha_m)_{|m|=M} = \left(\delta^m (\mathcal{L}^\alpha)^{-M/2}\Pi_0\right)_{|m|=M}.$$

This formal definition makes sense for all polynomials in $\mathbb{R}^d_+$ and if $M = 1$ it coincides with that given in Chapter 2. Another possible definition of higher order Riesz-Laguerre transforms, which could be taken under consideration, is

$$\mathcal{K}^{\alpha,M} = (K^\alpha_m)_{|m|=M} = \left(x^{m/2}\partial^m (\mathcal{L}^\alpha)^{-M/2}\Pi_0\right)_{|m|=M}, \tag{3.1}$$

with the notation $m/2 = (m_1/2, \dots, m_d/2)$. Nevertheless, more appropriate seems to be the first one since it better fits to a general framework and is more suitable from the transference point of view, whereas L^p boundedness of the operators K^α_m cannot be concluded by transference for all half-integer α (see Remark 3.4.9).

We will study L^p mapping properties of $\mathcal{R}^{\alpha,M}$ by exploiting analogous results for Hermite polynomial expansions. The setting is as follows. To the end of this chapter we assume, without further mention, that

$$\alpha_i = \frac{n_i}{2} - 1, \qquad n_i \in \mathbb{N}\backslash\{0\}, \quad i = 1, \dots, d.$$

Consider the space $\mathbb{R}^{|n|}$, where the multi-index $n = (n_1, \dots, n_d)$ is determined by α, as above. Notice, that $|n| \geq d$. Let

$$L^H = -\Delta/2 + x \cdot \nabla$$

be the Ornstein-Uhlenbeck operator in $\mathbb{R}^{|n|}$ and by $d\gamma(x) = \pi^{-|n|/2}\exp(-|x|^2)dx$ denote the associated Gaussian measure. For $x \in \mathbb{R}^{|n|}$ we write $x = (x^1, \dots, x^d)$, where $x^i = (x^i_1, \dots, x^i_{n_i})$ is the "block" of $\mathbb{R}^{|n|}$ corresponding to α_i. Thus x^i_j denotes the $(n_1 + \dots + n_{i-1} + j)$-th coordinate of x.

Let $\partial_{i,j} = \partial_{x^i_j}$. Hermite partial derivatives are defined by (cf. [**GIT**])

$$\partial^H_{i,j} = \frac{1}{\sqrt{2}}\partial_{i,j}, \qquad i = 1, \dots, d, \quad j = 1, \dots, n_i,$$

so that we have

$$\sum_{i=1}^{d}\sum_{j=1}^{n_i} (\partial^H_{i,j})^* \partial^H_{i,j} = L^H,$$

where

$$\left(\partial^H_{i,j}\right)^* = -\partial^H_{i,j} + \sqrt{2}x^i_j$$

is the formal adjoint of $\partial^H_{i,j}$ in $L^2(\mathbb{R}^{|n|}, d\gamma)$. To define higher order Riesz-Hermite transforms we introduce multi multi-indices $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_d) \in \mathbb{N}^{|n|}$, where each coordinate is also a multi-index: $\tilde{m}_i = (\tilde{m}_{i,1}, \dots, \tilde{m}_{i,n_i}) \in \mathbb{N}^{n_i}$. In the sequel multi multi-indices and their coordinates will always be distinguished by tildes, and will always refer to the Hermite setting. We denote $|\tilde{m}| = \sum_i |\tilde{m}_i| = \sum_{i,j} \tilde{m}_{i,j}$. According to the standard notation, the meaning of $(\partial^H)^{\tilde{m}_i}$ and $(\partial^H)^{\tilde{m}}$ is the following:

$$\begin{aligned}
\left(\partial^H\right)^{\tilde{m}_i} &= \left(\partial^H_{i,1}\right)^{\tilde{m}_{i,1}} \cdots \left(\partial^H_{i,n_i}\right)^{\tilde{m}_{i,n_i}}, \qquad i = 1, \dots, d, \\
\left(\partial^H\right)^{\tilde{m}} &= \left(\partial^H\right)^{\tilde{m}_1} \cdots \left(\partial^H\right)^{\tilde{m}_d}.
\end{aligned}$$

Similarly are defined the operators $\partial^{\tilde{m}_i}$ and $\partial^{\tilde{m}}$.

Let Π_0^H be the orthogonal projection onto the orthogonal complement of the eigenspace corresponding to the Hermite eigenvalue 0. Then the (normalized) Riesz-Hermite transform of order M is given by

$$\begin{aligned}\mathcal{R}^{H,M} &= \left(R^H_{\tilde{m}}\right)_{|\tilde{m}|=M} = \left(\Psi(\tilde{m})(\partial^H)^{\tilde{m}}(L^H)^{-M/2}\Pi_0^H\right)_{|\tilde{m}|=M}\\ &= \left[\Psi(\tilde{m})\left((\partial^H)^{\tilde{m}_1}(L^H)^{-M/2}\Pi_0^H,\dots,(\partial^H)^{\tilde{m}_d}(L^H)^{-M/2}\Pi_0^H\right)\right]_{|\tilde{m}|=M},\end{aligned}$$

where $\Psi(\tilde{m}) = \prod_{i=1}^d \left(\frac{|\tilde{m}_i|!}{\tilde{m}_i!}\right)^{1/2}$ with $\tilde{m}_i! = \prod_{j=1}^{n_i}(\tilde{m}_{i,j})!$, $i = 1,\dots,d$. Note, that the normalizing factor $\Psi(\tilde{m})$ may be estimated from above by $\sqrt{|\tilde{m}|!}$.

Riesz-Hermite transforms were studied with great intensity. The proof of L^p, $1 < p < \infty$, boundedness valid for any order and any dimension was found by P.A. Meyer, by probabilistic methods. Another proofs, in various cases, were given by Gundy [**G**], Pisier [**Pi**], Urbina [**Ur**], Gutierréz [**Gu**], and Gutierréz et al. [**GST**]. The corresponding weak type 1–1 was investigated by Fabes et al. [**FGS**], Forzani and Scotto [**FoSc**], and in [**GMST2**], see the survey [**Sj2**] for more details.

Recall that in Chapter 1 the quadratic transformation $\phi\colon \mathbb{R}^{|n|} \longrightarrow \mathbb{R}^d_+$ establishing a connection between Hermite and Laguerre function expansions was defined by (see (1.30))

$$\phi(x^1,\dots,x^d) = (|x^1|^2,\dots,|x^d|^2).$$

The transformation ϕ relates also Hermite polynomial setting in $\mathbb{R}^{|n|}$ and Laguerre polynomial setting in $\mathbb{R}^d_+$, which allows to transfer certain results from Hermite to Laguerre expansions. For instance, the following lemma shows a relation between L^H and $\mathcal{L}^\alpha$. The proof is an easy consequence of [**GIT**, Lemma 1.1].

LEMMA 3.2.1. *Let f be a polynomial in $\mathbb{R}^d_+$. Then, given $\xi \in \mathbb{R}$, we have*

$$\left(L^H\right)^\xi \Pi_0^H(f\circ\phi) = 2^\xi\left[(\mathcal{L}^\alpha)^\xi \Pi_0 f\right]\circ\phi.$$

In particular, $\Pi_0^H(f\circ\phi) = (\Pi_0 f)\circ\phi$.

In what follows we shall use the function $\varphi\colon \mathbb{N}^{|n|} \longrightarrow \mathbb{N}^d$ given by

$$\varphi(\tilde{m}) = (|\tilde{m}_1|,\dots,|\tilde{m}_d|).$$

Further, for a multi-index $m \in \mathbb{N}^d$, we define

$$\mathcal{A}(m) = \{\tilde{m} \in \mathbb{N}^{|n|} : \varphi(\tilde{m}) = m\}.$$

Observe that for $M \in \mathbb{N}$

$$\{\tilde{m} \in \mathbb{N}^{|n|} : |\tilde{m}| = M\} = \bigcup_{|m|=M} \mathcal{A}(m),$$

the summands being mutually disjoint.

3.3. Riesz-Laguerre transforms of order 2 and 3; explicit computations

Before passing to the general case, it is convenient to go through some computations, which provide a better insight in further development and also exhibit places, where difficulties appear.

We shall compute transference inequalities for Riesz-Laguerre transforms of order 2 and 3. Note that for the 1-st order Riesz transforms, as well as for other objects considered in [**GIT**], Laguerre and Hermite counterparts are related through equalities. This, however, is not the case of higher order Riesz operators.

3.3.1. Second order Riesz transforms; explicit computations.

PROPOSITION 3.3.1. *Let f be a polynomial in $\mathbb{R}^d_+$ and fix a multi-index m, such that $|m| = 2$. Then the following transference inequality holds:*

$$\left| [R^\alpha_m f] \circ \phi \right| \leq \left| \left(R^H_{\tilde{m}}(f \circ \phi) \right)_{\tilde{m} \in \mathcal{A}(m)} \right|_{\ell_2}.$$

PROOF. Without any loss of generality we will consider only two cases: $m = (1,1,0,\dots,0)$ and $m = (2,0,\dots,0)$ (of course the first case makes sense only if $d > 1$). Assume that $f \in \mathcal{H}_0^\perp$ and denote $\partial_i = \partial_{y_i}$, $i = 1,\dots,d$.

CASE 1. Let $m = (1,1,0,\dots,0)$. Using Lemma 3.2.1 we obtain

$$\begin{aligned}
\left| \left(R^H_{\tilde{m}}(f \circ \phi)(x) \right)_{\tilde{m} \in \mathcal{A}(m)} \right|^2_{\ell_2} &= \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \left(\partial^H_{1,j_1} \partial^H_{2,j_2} (L^H)^{-1} (f \circ \phi)(x) \right)^2 \\
&= 2^{-4} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \left(\partial_{1,j_1} \partial_{2,j_2} \left(\left[(\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right) \right)^2 \\
&= 2^{-4} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \left(2x^1_{j_1} 2x^2_{j_2} \left[\partial_1 \partial_2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2 \\
&= \sum_{j_1=1}^{n_1} (x^1_{j_1})^2 \sum_{j_2=1}^{n_2} (x^2_{j_2})^2 \left(\left[\partial_1 \partial_2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2 \\
&= |x^1|^2 |x^2|^2 \left(\left[\partial_1 \partial_2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2 \\
&= \left(\left[\delta_1 \delta_2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2 \\
&= \left([R^\alpha_m f] \circ \phi(x) \right)^2.
\end{aligned}$$

CASE 2. Let $m = (2,0,\dots,0)$. Again by Lemma 3.2.1 we get

$$\begin{aligned}
\left| \left(R^H_{\tilde{m}}(f \circ \phi)(x) \right)_{\tilde{m} \in \mathcal{A}(m)} \right|^2_{\ell_2} &= \sum_{j,l=1}^{n_1} \left(\partial^H_{1,j} \partial^H_{1,l} (L^H)^{-1} (f \circ \phi)(x) \right)^2 \\
&= 2^{-4} \sum_{j,l=1}^{n_1} \left(\partial_{1,j} \partial_{1,l} \left(\left[(\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right) \right)^2 \\
&= 2^{-4} \sum_{\substack{j,l=1 \\ j \neq l}}^{n_1} \left(2x^1_j 2x^1_l \left[(\partial_1)^2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2
\end{aligned}$$

$$+ 2^{-4} \sum_{\substack{j=1 \\ l=j}}^{n_1} \left(2 \left[\partial_1 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) + 2x^1_j 2x^1_l \left[(\partial_1)^2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2$$

$$= \sum_{j,l=1}^{n_1} \left(x^1_j x^1_l \left[(\partial_1)^2 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2 + \sum_{j=1}^{n_1} \left(\frac{1}{2} \left[\partial_1 (\mathcal{L}^\alpha)^{-1} f \right] \circ \phi(x) \right)^2$$

$$+\sum_{j=1}^{n_1}(x_j^1)^2\left[(\partial_1)^2(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\left[\partial_1(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)$$

$$=\left(|x^1|^2\left[(\partial_1)^2(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\right)^2+|x^1|^2\left[(\partial_1)^2(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\left[\partial_1(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)$$

$$+\left(\frac{1}{2}\left[\partial_1(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\right)^2+\frac{n_1-1}{4}\left(\left[\partial_1(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\right)^2$$

$$\geq\left(|x^1|^2\left[(\partial_1)^2(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)+\frac{1}{2}\left[\partial_1(\mathcal{L}^\alpha)^{-1}f\right]\circ\phi(x)\right)^2=([R_m^\alpha f]\circ\phi(x))^2.$$

The last equality follows since

$$(\delta_1)^2=y_1(\partial_1)^2+\frac{1}{2}\partial_1.$$

□

3.3.2. Third order Riesz transforms; explicit computations.

PROPOSITION 3.3.2. *Let f be a polynomial in $\mathbb{R}_+^d$ and fix a multi-index m such that $|m|=3$. Then*

$$\left|\,[R_m^\alpha f]\circ\phi\right|\leq\left|\left(R_{\tilde m}^H(f\circ\phi)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2}. \tag{3.2}$$

PROOF. Assume that $f\in\mathcal{H}_0^\perp$. Without any loss of generality we will consider only three cases.

CASE 1. If $d\geq 3$ and $m=(1,1,1,0,\dots,0)$ then (3.2) is verified exactly in the same way as for $m=(1,1,0,\dots,0)$. Thus the computation is omitted.

CASE 2. Assume that $d\geq 2$ and let $m=(2,1,0,\dots,0)$. Using Lemma 3.2.1 we write

$$\left|\left(R_{\tilde m}^H(f\circ\phi)(x)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2}^2=\sum_{j_1,l_1=1}^{n_1}\sum_{j_2=1}^{n_2}\left(\partial_{1,j_1}^H\partial_{1,l_1}^H\partial_{2,j_2}^H(L^H)^{-3/2}(f\circ\phi)(x)\right)^2$$

$$=2^{-6}\sum_{j_1,l_1=1}^{n_1}\sum_{j_2=1}^{n_2}\left(\partial_{1,j_1}\partial_{1,l_1}\partial_{2,j_2}\left(\left[(\mathcal{L}^\alpha)^{-3/2}f\right]\circ\phi(x)\right)\right)^2$$

$$=2^{-4}\sum_{j_2=1}^{n_2}(x_{j_2}^2)^2\sum_{j_1,l_1=1}^{n_1}\left(\partial_{1,j_1}\partial_{1,l_1}\left(\left[\partial_2(\mathcal{L}^\alpha)^{-3/2}f\right]\circ\phi(x)\right)\right)^2.$$

Now we may estimate the internal sum in the same way as in the case $m=(2,0,\dots,0)$, see the proof of Proposition 3.3.1. Hence the last expression is further bounded by

$$|x^2|^2\left(|x^1|^2\left[(\partial_1)^2\partial_2(\mathcal{L}^\alpha)^{-3/2}f\right]\circ\phi(x)+\frac{1}{2}\left[\partial_1\partial_2(\mathcal{L}^\alpha)^{-3/2}f\right]\circ\phi(x)\right)^2$$
$$=([R_m^\alpha f]\circ\phi(x))^2,$$

since

$$(\delta_1)^2\delta_2=y_1\sqrt{y_2}(\partial_1)^2\partial_2+\frac{1}{2}\sqrt{y_2}\partial_1\partial_2.$$

CASE 3. Let $m = (3, 0, \dots, 0)$. Assume that $F\colon \mathbb{R}^{|n|} \longrightarrow \mathbb{R}^d_+$ is sufficiently regular. Then a straightforward computation gives

$$\partial_{1,j}\partial_{1,l}\partial_{1,k}\left[F \circ \phi(x)\right] =$$
$$2^3\left[x^1_j x^1_l x^1_k[(\partial_1)^3 F]\circ\phi(x) + \frac{1}{2}\left(x^1_j\chi_{\{l=k\}} + x^1_l\chi_{\{j=k\}} + x^1_k\chi_{\{j=l\}}\right)[(\partial_1)^2 F]\circ\phi(x)\right],$$

where $\chi_{\{l=k\}} = 1$ if $l = k$ and $\chi_{\{l=k\}} = 0$ otherwise. Therefore denoting

$$\mathcal{I} = [(\partial_1)^3(L^H)^{-3/2}]\circ\phi(x) \qquad \text{and} \qquad \mathcal{J} = \frac{1}{2}[(\partial_1)^2(L^H)^{-3/2}]\circ\phi(x),$$

and making use of Lemma 3.2.1 we obtain

$$\left|\left(R^H_{\tilde m}(f\circ\phi)(x)\right)_{\tilde m\in\mathcal{A}(m)}\right|^2_{\ell_2} = \sum_{j,l,k=1}^{n_1}\left(\partial^H_{1,j}\partial^H_{1,l}\partial^H_{1,k}(L^H)^{-3/2}(f\circ\phi)(x)\right)^2$$
$$= \sum_{\substack{j,l,k=1\\ j\neq l\neq k\neq j}}^{n_1}(x^1_j x^1_l x^1_k\mathcal{I})^2 + \sum_{\substack{j,l,k=1\\ j=l\neq k}}^{n_1}(x^1_j x^1_l x^1_k\mathcal{I} + x^1_k\mathcal{J})^2 + \sum_{\substack{j,l,k=1\\ j\neq l=k}}^{n_1}(x^1_j x^1_l x^1_k\mathcal{I} + x^1_j\mathcal{J})^2$$
$$+ \sum_{\substack{j,l,k=1\\ j=k\neq l}}^{n_1}(x^1_j x^1_l x^1_k\mathcal{I} + x^1_l\mathcal{J})^2 + \sum_{\substack{j,l,k=1\\ j=k=l}}^{n_1}(x^1_j x^1_l x^1_k\mathcal{I} + (x^1_j + x^1_l + x^1_k)\mathcal{J})^2$$
$$= \mathcal{I}^2\sum_{j,l,k=1}^{n_1}(x^1_j x^1_l x^1_k)^2 + \sum_{\substack{j,l,k=1\\ j=l}}^{n_1}\left[2\mathcal{I}\mathcal{J}(x^1_j)^2(x^1_k)^2 + \mathcal{J}^2(x^1_k)^2\right]$$
$$+ \sum_{\substack{j,l,k=1\\ l=k}}^{n_1}\left[2\mathcal{I}\mathcal{J}(x^1_j)^2(x^1_l)^2 + \mathcal{J}^2(x^1_j)^2\right] + \sum_{\substack{j,l,k=1\\ j=k}}^{n_1}\left[2\mathcal{I}\mathcal{J}(x^1_l)^2(x^1_k)^2 + \mathcal{J}^2(x^1_l)^2\right]$$
$$+ \sum_{\substack{j,l,k=1\\ j=k=l}}^{n_1}6\mathcal{J}^2(x^1_j)^2 = \mathcal{I}^2|x^1|^6 + 6\mathcal{I}\mathcal{J}|x^1|^4 + (3n_1+6)\mathcal{J}^2|x^1|^2$$
$$\geq \mathcal{I}^2|x^1|^6 + 6\mathcal{I}\mathcal{J}|x^1|^4 + 9\mathcal{J}^2|x^1|^2 = \left(\mathcal{I}|x^1|^3 + 3|x^1|\mathcal{J}\right)^2$$
$$= \left(|x^1|^3\,[(\partial_1)^3(L^H)^{-3/2}]\circ\phi(x) + \frac{3}{2}|x^1|\,[(\partial_1)^2(L^H)^{-3/2}]\circ\phi(x)\right)^2$$
$$= ([R^\alpha_m f]\circ\phi(x))^2.$$

The last equality holds, because

$$(\delta_1)^3 = (y_1)^{3/2}(\partial_1)^3 + \frac{3}{2}\sqrt{y_1}(\partial_1)^2.$$

□

3.4. Higher order Riesz-Laguerre transforms; the general case

The main objective of this section is to prove the transference inequality for Riesz-Laguerre transforms of arbitrary finite order, which is stated in Theorem 3.4.6

below. We start by defining the coefficients

$$E_{N,k} = 2^{N-2k}\frac{N!}{k!(N-2k)!}, \qquad N = 0,1,2,\dots, \quad 0 \le k \le N/2.$$

Note, that the Hermite polynomials express explicitly by (cf. [**Le**, (4.17.2)])

$$H_N(x) = \sum_{0\le k\le N/2} (-1)^k E_{N,k}\, x^{N-2k}, \qquad N = 0,1,2,\dots,$$

hence $E_{N,k}$ is the absolute value of the coefficient standing at the $(N-2k)$-th power in the N-th Hermite polynomial.

LEMMA 3.4.1. *Given $N \in \mathbb{N}$ and a sufficiently smooth function $g\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ we have*

$$\partial^N \left[g(y^2)\right] = 2^N \left[\delta^N g\right](y^2), \tag{3.3}$$

$$\partial^N \left[g(y^2)\right] = \sum_{0\le k\le N/2} E_{N,k}\, y^{N-2k} \left[\partial^{N-k} g\right](y^2), \tag{3.4}$$

$$\delta^N g(y) = 2^{-N} \sum_{0\le k\le N/2} E_{N,k}\, \sqrt{y}^{\,N-2k} \left[\partial^{N-k} g\right](y). \tag{3.5}$$

PROOF. The identity (3.3) is easily verified by induction. (3.4) follows by Faá di Bruno's formula for the N-th derivative of the composition of two functions:

$$\partial^N (g\circ f)(t) = N! \sum_{i=1}^{N} (\partial^i g)\circ f(t) \sum_{\substack{k\in\mathbb{N}^N,\ |k|=i \\ k_1+2k_2+\ldots+Nk_N=N}} \prod_{j=1}^{N} \frac{1}{k_j!}\left(\frac{\partial^j f(t)}{j!}\right)^{k_j}.$$

Finally, (3.5) is a consequence of the two previous identities. □

COROLLARY 3.4.2. *Let $F\colon \mathbb{R}^d_+ \longrightarrow \mathbb{R}^d$ be sufficiently regular. Given multi-indices $\tilde{m} \in \mathbb{N}^{|n|}$ and $m \in \mathbb{N}^d$, we have*

$$\left(\partial^H\right)^{\tilde{m}} \left[F\circ\phi\right](x) = 2^{-|\tilde{m}|/2} \sum_{\bar{0}\le\tilde{k}\le\tilde{m}/2} E_{\tilde{m},\tilde{k}}\, x^{\tilde{m}-2\tilde{k}} \left[\partial^{\varphi(\tilde{m})-\varphi(\tilde{k})} F\right]\circ\phi(x),$$

$$\left[\delta^m F\right]\circ\phi(x) = 2^{-|m|} \sum_{\bar{0}\le k\le m/2} E_{m,k}\, [\phi(x)]^{m/2-k} \left[\partial^{m-k} F\right]\circ\phi(x).$$

In particular, if f is a polynomial in $\mathbb{R}^d_+$, then denoting $F = (\mathcal{L}^\alpha)^{-|m|/2}\Pi_0 f$ and using Lemma 3.2.1 we get

$$R^H_{\tilde{m}}(f\circ\phi)(x) = 2^{-|\tilde{m}|}\Psi(\tilde{m}) \sum_{\bar{0}\le\tilde{k}\le\tilde{m}/2} E_{\tilde{m},\tilde{k}}\, x^{\tilde{m}-2\tilde{k}} \left[\partial^{\varphi(\tilde{m})-\varphi(\tilde{k})} F\right]\circ\phi(x),$$

$$\left[R^\alpha_m f\right]\circ\phi(x) = 2^{-|m|} \sum_{\bar{0}\le k\le m/2} E_{m,k}\, [\phi(x)]^{m/2-k} \left[\partial^{m-k} F\right]\circ\phi(x).$$

In the above corollary the following notation has been used:

$$E_{\tilde{m},\tilde{k}} = \prod_{i=1}^{d}\prod_{j=1}^{n_i} E_{\tilde{m}_{i,j},\tilde{k}_{i,j}}, \qquad\qquad E_{m,k} = \prod_{i=1}^{d} E_{m_i,k_i},$$

$$x^{\tilde{m}} = \prod_{i=1}^{d}\prod_{j=1}^{n_i}(x_j^i)^{\tilde{m}_{i,j}}, \qquad (\phi(x))^m = \prod_{i=1}^{d}|x^i|^{2m_i},$$

$$\begin{aligned} \bar{0} \le \tilde{k} \le \tilde{m} \quad &\equiv \quad 0 \le \tilde{k}_{i,j} \le \tilde{m}_{i,j}, \quad 1 \le i \le d, \ 1 \le j \le n_i, \\ \bar{0} \le k \le m \quad &\equiv \quad 0 \le k_i \le m_i, \quad 1 \le i \le d, \end{aligned}$$

where $\bar{0}$ denotes the multi-dimensional zero $(0, \dots, 0) \in \mathbb{N}^N$, the value of N being dependent on the particular context. Such notation will be used in the sequel without further comments or explanations.

The following technical lemma is crucial. It enables us to establish a relation between higher order Hermite derivatives in $\mathbb{R}^{|n|}$ and higher order Laguerre derivatives in $\mathbb{R}^d_+$ (see Proposition 3.4.10 below). Recall that the Pochhammer symbol $(\lambda)_i = 1$ if $i = 0$ and $(\lambda)_i = \lambda(\lambda+1)\cdots(\lambda+i-1)$ for $i = 1, 2, \dots$.

LEMMA 3.4.3. *Let $m \in \mathbb{N}^d$ be a fixed multi-index. Then, for an arbitrary set of complex numbers $\{\beta_j\}_{\bar{0}\le j\le m/2}$ and all $x \in \mathbb{R}^{|n|}$, we have*

$$(3.6) \quad \sum_{\tilde{m}\in\mathcal{A}(m)} \frac{m!}{\tilde{m}!}\Big(\sum_{0\le\tilde{k}\le\tilde{m}/2} E_{\tilde{m},\tilde{k}}\, x^{\tilde{m}-2\tilde{k}}\beta_{\varphi(\tilde{k})}\Big)^2$$
$$= \sum_{\bar{0}\le j\le m/2}\left[\frac{2^{2|j|}}{j!}(-m)_{2j}\big((n-\bar{1})/2\big)_j\Big(\sum_{\bar{0}\le k\le m/2-j} E_{m-2j,k}[\phi(x)]^{m/2-j-k}\beta_{k+j}\Big)^2\right],$$

where $m! = m_1!\cdots m_d!$, $\bar{1} = (1, \dots, 1) \in \mathbb{N}^d$ and $(\lambda)_j = (\lambda_1)_{j_1}\cdots(\lambda_d)_{j_d}$ for $\lambda \in \mathbb{R}^d$ and a multi-index $j \in \mathbb{N}^d$.

PROOF. After expanding both sides, the identity (3.6) takes the form

$$(3.7) \quad \sum_{\tilde{m}\in\mathcal{A}(m)} \frac{m!}{\tilde{m}!} \sum_{\substack{\bar{0}\le\tilde{k}\le\tilde{m}/2 \\ \bar{0}\le\tilde{l}\le\tilde{m}/2}} E_{\tilde{m},\tilde{k}}E_{\tilde{m},\tilde{l}}\, x^{2\tilde{m}-2\tilde{k}-2\tilde{l}}\beta_{\varphi(\tilde{k})}\beta_{\varphi(\tilde{l})} =$$
$$\sum_{\bar{0}\le j\le m/2} \frac{2^{2|j|}}{j!}\Big(\frac{n-\bar{1}}{2}\Big)_j (-m)_{2j} \sum_{\substack{\bar{0}\le k\le m/2-j \\ \bar{0}\le l\le m/2-j}} E_{m-2j,k}E_{m-2j,l}[\phi(x)]^{m-2j-k-l}\beta_{k+j}\beta_{l+j}.$$

It remains to show that the coefficients of $\beta_k\beta_l$, $\bar{0} \le k, l \le m/2$, coincide on both sides. To do this, let us introduce the polynomials

$$g_N(2u, p) = \sum_{0\le k\le N/2} E_{N,k}\, p^k u^{N-2k}, \qquad u, p \in \mathbb{R},$$

which are a generalization of Hermite polynomials, because $H_N(u) = g_N(2u, -1)$. Noteworthy, $g_N(u, p)$ are contained in certain general classes of polynomials considered by Brafman, Gould and Hopper, and others, see [**SrMa**] and references therein. Observe that

$$g_N(2u, p) = \Big(\frac{i}{\sqrt{p}}\Big)^{-N} H_N\Big(\frac{i}{\sqrt{p}}u\Big), \qquad p \ne 0,$$

hence, in virtue of classical Mehler's formula which holds also with imaginary arguments (this seems to be well-known and may be proved by essentially the same reasoning as in the case of real arguments, see the proof in [**Sj2**]) we get

$$\begin{aligned}\sum_{N=0}^{\infty} g_N(2u,p)g_N(2v,q)\frac{z^N}{2^N N!} &= \sum_{N=0}^{\infty} H_N\left(\frac{i}{\sqrt{p}}u\right)H_N\left(\frac{i}{\sqrt{q}}v\right)\frac{(-\sqrt{pq}z)^N}{2^N N!} \\ &= \frac{1}{\sqrt{1-pqz^2}}\exp\left(\frac{z^2(qu^2+pv^2)+2zuv}{1-pqz^2}\right),\end{aligned}$$

provided $|z| < |pq|^{-1/2}$. In particular, for $u = v$ we have

$$\sum_{N=0}^{\infty} g_N(2u,p)g_N(2u,q)\frac{z^N}{2^N N!} = \frac{1}{\sqrt{1-pqz^2}}\exp\left(u^2\frac{(p+q)z^2+2z}{1-pqz^2}\right). \tag{3.8}$$

Next, consider the multi-dimensional polynomials

$$g_{\tilde{m}_i}(2x^i,p_i) = \prod_{j=1}^{n_i} g_{\tilde{m}_{i,j}}(2x^i_j,p_i), \qquad p_i \in \mathbb{R}, \quad i = 1,\dots,d.$$

Using the generating formula (3.8) we obtain

$$\begin{aligned}&\sum_{\tilde{m}_i\in\mathbb{N}^{n_i}} g_{\tilde{m}_i}(2x^i,p_i)g_{\tilde{m}_i}(2x^i,q_i)\frac{z_i^{|\tilde{m}_i|}}{2^{|\tilde{m}_i|}\tilde{m}_i!} \\ &\qquad = \left(1-p_iq_iz_i^2\right)^{-n_i/2}\exp\left(|x^i|^2\frac{(p_i+q_i)z_i^2+2z_i}{1-p_iq_iz_i^2}\right),\end{aligned}$$

the identity being valid if $|z_i| < |p_iq_i|^{-1/2}$, $i = 1,\dots,d$. Thus, denoting $z = (z_1,\dots,z_d)$, $p = (p_1,\dots,p_d)$, $q = (q_1,\dots,q_d)$, we have

$$\begin{aligned}&\sum_{\tilde{m}\in\mathbb{N}^{|n|}} g_{\tilde{m}_1}(2x^1,p_1)g_{\tilde{m}_1}(2x^1,q_1)\cdots g_{\tilde{m}_d}(2x^d,p_d)g_{\tilde{m}_d}(2x^d,q_d)\frac{z^{\varphi(\tilde{m})}}{2^{|\tilde{m}|}\tilde{m}!} \\ &\qquad = \prod_{i=1}^{d}\sum_{\tilde{m}_i\in\mathbb{N}^{n_i}} g_{\tilde{m}_i}(2x^i,p_i)g_{\tilde{m}_i}(2x^i,q_i)\frac{z_i^{|\tilde{m}_i|}}{2^{|\tilde{m}_i|}\tilde{m}_i!} \\ &\qquad = \prod_{i=1}^{d}\left(1-p_iq_iz_i^2\right)^{-n_i/2}\exp\left(|x^i|^2\frac{(p_i+q_i)z_i^2+2z_i}{1-p_iq_iz_i^2}\right) \\ &\qquad = \left(\sum_{j\in\mathbb{N}^d}\frac{1}{j!}\left(\frac{n-\bar{1}}{2}\right)_j p^jq^jz^{2j}\right)\left(\sum_{k\in\mathbb{N}^d}\prod_{i=1}^{d} g_{k_i}(2|x^i|,p_i)g_{k_i}(2|x^i|,q_i)\frac{z^{k_i}}{2^{k_i}k_i!}\right),\end{aligned}$$

since

$$(1-u)^{-(N-1)/2} = \sum_{i=0}^{\infty}\left((N-1)/2\right)_i\frac{u^i}{i!}, \qquad N = 1,2,\dots, \quad |u| < 1.$$

Now, comparing the coefficients of z^m we see that

$$\begin{aligned}&\sum_{\tilde{m}\in\mathcal{A}(m)}\frac{m!}{\tilde{m}!}g_{\tilde{m}_1}(2x^1,p_1)g_{\tilde{m}_1}(2x^1,q_1)\cdots g_{\tilde{m}_d}(2x^d,p_d)g_{\tilde{m}_d}(2x^d,q_d) \\ &= \sum_{\bar{0}\le j\le m/2}\frac{2^{2|j|}}{j!}(-m)_{2j}\left((n-1)/2\right)_j p^jq^j\prod_{i=1}^{d} g_{m_i-2j_i}(2|x^i|,p_i)\; g_{m_i-2j_i}(2|x^i|,q_i),\end{aligned}$$

which after expansions yields

$$\sum_{\tilde{m}\in\mathcal{A}(m)} \frac{m!}{\tilde{m}!} \sum_{\substack{\bar{0}\le\tilde{k}\le\tilde{m}/2\\ \bar{0}\le\tilde{l}\le\tilde{m}/2}} E_{\tilde{m},\tilde{k}} E_{\tilde{m},\tilde{l}} x^{2\tilde{m}-2\tilde{k}-2\tilde{l}} p^{\varphi(\tilde{k})} q^{\varphi(\tilde{l})} =$$

$$\sum_{\bar{0}\le j\le m/2} \frac{2^{2|j|}}{j!}(-m)_{2j}\big((n-\bar{1})/2\big)_j \sum_{\substack{\bar{0}\le k\le m/2-j\\ \bar{0}\le l\le m/2-j}} E_{m-2j,k}E_{m-2j,l}[\phi(x)]^{m-2j-k-l}p^{k+j}q^{l+j}.$$

Since the above identity holds at least for all $p, q \in \mathbb{R}^d_+$, it follows that the coefficients of $p^k q^l$, $\bar{0} \le k, l \le m/2$, are equal on both sides. This, in view of (3.7), finishes the proof. □

REMARK 3.4.4. As an interesting application of the technique exploited in the above proof one may derive bilinear composition formulas for Hermite and Gould-Hopper polynomials, see [**GrNo**]. For example, it follows that

$$\sum_{|m|=M} \frac{1}{m!}\big[H_m(x)\big]^2 = \sum_{0\le j\le M/2} \frac{2^{2j}}{j!(M-2j)!}\left(\frac{d-1}{2}\right)_j \big[H_{M-2j}(|x|)\big]^2,$$

for all $M \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This is the simplest formula of those discussed in [**GrNo**], nevertheless it seems to be new.

LEMMA 3.4.5. *Let f be a polynomial in $\mathbb{R}^d_+$. For all $m \in \mathbb{N}^d$ and $x \in \mathbb{R}^{|n|}$ we have*

$$(3.9)\quad \Big|\big(R^H_{\tilde{m}}(f\circ\phi)(x)\big)_{\tilde{m}\in\mathcal{A}(m)}\Big|^2_{\ell_2} = 2^{-|m|} \sum_{\bar{0}\le j\le m/2} E_{m,j}\big((n-\bar{1})/2\big)_j \Big(\big[\delta^{m-2j}\partial^j F\big]\circ\phi(x)\Big)^2,$$

where $F = (\mathcal{L}^\alpha)^{-|m|/2}\Pi_0 f$.

PROOF. By Corollary 3.4.2 and Lemma 3.4.3

$$\begin{aligned}
&\Big|\big(R^H_{\tilde{m}}(f\circ\phi)(x)\big)_{\tilde{m}\in\mathcal{A}(m)}\Big|^2_{\ell_2}\\
&\quad = 2^{-2|m|}\sum_{\tilde{m}\in\mathcal{A}(m)}\frac{m!}{\tilde{m}!}\Big(\sum_{\bar{0}\le\tilde{k}\le\tilde{m}/2} E_{\tilde{m},\tilde{k}}x^{\tilde{m}-2\tilde{k}}\Big[\partial^{m-\varphi(\tilde{k})}F\Big]\circ\phi(x)\Big)^2\\
&\quad = 2^{-2|m|}\sum_{\bar{0}\le j\le m/2}\Bigg[\frac{2^{2|j|}}{j!}(-m)_{2j}\big((n-\bar{1})/2\big)_j\\
&\qquad\times\Big(\sum_{\bar{0}\le k\le m/2-j} E_{m-2j,k}[\phi(x)]^{m/2-j-k}\Big[\partial^{m-j-k}F\Big]\circ\phi(x)\Big)^2\Bigg],
\end{aligned}$$

which by (3.5) is equal to the RHS in (3.9). □

THEOREM 3.4.6. *Let f be a polynomial in $\mathbb{R}^d_+$. Then*

$$|[R^\alpha_m f]\circ\phi| \le \Big|\big(R^H_{\tilde{m}}(f\circ\phi)\big)_{\tilde{m}\in\mathcal{A}(m)}\Big|_{\ell_2}$$

for any multi-index $m \in \mathbb{N}^d$.

PROOF. In view of Lemma 3.4.5 and the definition of R_m^α we have

$$\left|\left(R_{\tilde m}^H(f\circ\phi)(x)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2}^2-([R_m^\alpha f]\circ\phi(x))^2 \tag{3.10}$$
$$=2^{-|m|}\sum_{\substack{\bar 0\le j\le m/2\\ j\neq\bar 0}}\left((n-\bar 1)/2\right)_j E_{m,j}\left(\left[\delta^{m-2j}\partial^j F\right]\circ\phi(x)\right)^2\ \ge 0$$

for all $x\in\mathbb{R}^{|n|}$. □

REMARK 3.4.7. If $\alpha=(-1/2,\dots,-1/2)$, which is equivalent to $n=(1,\dots,1)$, then a stronger result holds than that stated in Theorem 3.4.6, namely

$$[R_m^\alpha f]\circ\phi=R_m^H(f\circ\phi),$$

for all polynomials f in $\mathbb{R}_+^d$ and an arbitrary multi-index $m\in\mathbb{N}^d$. Moreover, for any half-integer α, the first order Riesz transforms satisfy

$$\left|[R_{e_i}^\alpha f]\circ\phi\right|=\left|(R_{\tilde m}^H(f\circ\phi))_{\tilde m\in\mathcal{A}(e_i)}\right|_{\ell_2},\qquad i=1,\dots,d.$$

The first of the above facts follows by Corollary 3.4.2. The other one is a direct consequence of (3.10).

Let us also consider the operators

$$K_{m,j}^\alpha=x^{m/2-j}\partial^{m-j}(\mathcal{L}^\alpha)^{-|m|/2}\Pi_0,\qquad \bar 0\le j\le m/2,\quad m\in\mathbb{N}^d,$$

which are closely related to the Riesz-Laguerre transform of order $|m|$, because R_m^α is a linear combination of $K_{m,j}^\alpha$, $\bar 0\le j\le m/2$. These operators turned out to be important in studying Sobolev spaces associated with Laguerre polynomial expansions (see [**GLLNU**]). Note that $K_{m,\bar 0}^\alpha$ coincides with K_m^α in (3.1).

THEOREM 3.4.8. *Assume that* $\min_i\alpha_i>-1/2$ *and let* f *be a polynomial in* $\mathbb{R}_+^d$. *Then, for all multi-indices* $m,j\in\mathbb{N}^d$ *such that* $\bar 0\le j\le m/2$

$$\left|[K_{m,j}^\alpha f]\circ\phi\right|\le C\left|\left(R_{\tilde m}^H(f\circ\phi)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2}, \tag{3.11}$$

with C *independent of* d *and* α.

PROOF. Define the operators

$$S_{m,j}^\alpha\ =\ \mathcal{B}(m,j,n)\sum_{j\le k\le m/2}E_{m-2j,k-j}K_{m,k}^\alpha,\qquad \bar 0\le j\le m/2, \tag{3.12}$$

where

$$\mathcal{B}(m,j,n)=\left(2^{-2|m|}\frac{2^{2|j|}}{j!}(-m)_{2j}\left((n-\bar 1)/2\right)_j\right)^{1/2}.$$

By the proof of Lemma 3.4.5 we have

$$\left|\left(R_{\tilde m}^H(f\circ\phi)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2}^2=\sum_{\bar 0\le j\le m/2}\left[(S_{m,j}^\alpha f)\circ\phi\right]^2.$$

Thus

$$\left|(S_{m,j}^\alpha f)\circ\phi\right|\le\left|\left(R_{\tilde m}^H(f\circ\phi)\right)_{\tilde m\in\mathcal{A}(m)}\right|_{\ell_2},\qquad \bar 0\le j\le m/2.$$

When $j=m/2$ or $j=m/2-e_i/2$ then the sum in (3.12) has only one term and (3.11) follows for $K_{m,m/2}^\alpha$. When $j=m/2-e_i$ for $m_i\ge 2$, the sum in (3.12) has two terms, one of which is $K_{m,m/2}^\alpha$ considered in the first step, so (3.11) holds also for the second term. Iterating this procedure we get (3.11) for all the operators $K^{\alpha}_{m,j}$. The constant C is independent of n, because the Pochhammer symbol $\big((n-\bar{1})/2\big)_j$ is monotone in j. □

REMARK 3.4.9. The assumption about $\min_i \alpha_i$ is essential. If $\alpha_i = -1/2$ for some i, then $\big((n-\bar{1})/2\big)_j$ vanishes whenever $j_i > 0$ and hence the conclusion does not follow.

The result below may be regarded as a generalization of the formula (3.4), with the square function on $\mathbb{R}$ replaced by the transference transformation ϕ.

PROPOSITION 3.4.10. *Given a sufficiently smooth function f in $\mathbb{R}^d_+$ and a multi-index $m \in \mathbb{N}^d$, we have*

$$\begin{aligned}\sum_{\tilde{m}\in\mathcal{A}(m)} \frac{m!}{\tilde{m}!}\left[\partial^{\tilde{m}}(f\circ\phi)(x)\right]^2 &= \sum_{\bar{0}\le j\le m/2}\Bigg[\frac{2^{2|j|}}{j!}(-m)_{2j}\big((n-\bar{1})/2\big)_j \\ &\quad\times\Bigg(\sum_{\bar{0}\le k\le m/2-j} E_{m-2j,k}[\phi(x)]^{m/2-j-k}\left[\partial^{m-j-k}f\right]\circ\phi(x)\Bigg)^2\Bigg] \\ &= 2^{|m|}\sum_{\bar{0}\le j\le m/2} E_{m,j}\big((n-\bar{1})/2\big)_j\Big(\left[\delta^{m-2j}\partial^j f\right]\circ\phi(x)\Big)^2.\end{aligned}$$

PROOF. These identities were already obtained implicitly and are a consequence of Lemma 3.4.3, see also Lemma 3.4.5 and its proof. □

3.5. Conclusions; L^p boundedness and weak type 1-1

THEOREM 3.5.1. *Let $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_i = n_i/2-1$ and $n_i \in \mathbb{N}\backslash\{0\}$. Then, given $1 < p < \infty$ and $M \in \mathbb{N}$, we have*

$$\Big\| |\mathcal{R}^{\alpha,M} f|_{\ell^2}\Big\|_{L^p(\mathbb{R}^d_+, d\mu_\alpha)} \le C\|f\|_{L^p(\mathbb{R}^d_+, d\mu_\alpha)}, \tag{3.13}$$

for all polynomials f. Consequently, each R^{α}_m, $|m| = M$, extends uniquely to a bounded linear operator in $L^p(\mathbb{R}^d_+, d\mu_\alpha)$. Moreover, the constant C is independent of the dimension d and the type multi-index α.

PROOF OF THEOREM 3.5.1. By [**GIT**, Lemma 2.1] we have

$$\int_{\mathbb{R}^d_+} g(y)\, d\mu_\alpha(y) = \mathcal{C}_{d,n}\int_{R^{|n|}} g\circ\phi(x)\, d\gamma(x). \tag{3.14}$$

Hence, by Theorem 3.4.6 and P.A. Meyer's result [**Me**] on the $L^p(d\gamma)$ boundedness of higher order Riesz-Hermite transforms, for any polynomial f in $\mathbb{R}^d_+$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d_+} |\mathcal{R}^{\alpha,M} f(y)|^p_{\ell^2}\, d\mu_\alpha(y) &= \mathcal{C}_{d,n}\int_{R^{|n|}} |(\mathcal{R}^{\alpha,M} f)\circ\phi(x)|^p_{\ell^2}\, d\gamma(x) \\ &\le \mathcal{C}_{d,n}\int_{\mathbb{R}^{|n|}} |\mathcal{R}^{H,M}(f\circ\phi)(x)|^p_{\ell^2}\, d\gamma(x) \\ &\le C_{p,M}\,\mathcal{C}_{d,n}\int_{\mathbb{R}^{|n|}} |(f\circ\phi)(x)|^p\, d\gamma(x)\end{aligned}$$

$$= C_{p,M} \int_{\mathbb{R}^d_+} |f(y)|^p \, d\mu_\alpha(y).$$

Thus (3.13) is justified. □

Weak type 1–1 for Riesz-Laguerre transforms of order 1 was obtained for half-integer α's in [**GIT**], by the method of transference. Concerning the transforms of order 2, we have

THEOREM 3.5.2. *Let α be half-integer. Then the Riesz-Laguerre transforms R^α_m of order 2 extend uniquely to weakly bounded linear operators on $L^1(\mathbb{R}^d_+, d\mu_\alpha)$.*

PROOF. Assume that $|m| = 2$. Let f be a polynomial in $\mathbb{R}^d_+$ and denote

$$\begin{aligned} Y_\lambda &= \{y \in \mathbb{R}^d_+ : |R^\alpha_m f(y)| > \lambda\}, \\ X_\lambda &= \left\{x \in \mathbb{R}^{|n|} : \left| \left(R^H_{\tilde{m}}(f \circ \phi)(x)\right)_{\tilde{m} \in \mathcal{A}(m)} \right|_{\ell^2} > \lambda \right\}. \end{aligned}$$

Note, that by Theorem 3.4.6

$$\mathbf{1}_{Y_\lambda} \circ \phi \le \mathbf{1}_{X_\lambda},$$

where $\mathbf{1}_{Y_\lambda}$ and $\mathbf{1}_{X_\lambda}$ are the indicator functions of Y_λ and X_λ, respectively. Hence, using (3.14) and the weak type 1–1 results for the corresponding Riesz-Hermite transforms [**GMST2**] we get

$$\begin{aligned} \mu_\alpha(Y_\lambda) &= \int_{\mathbb{R}^d_+} \mathbf{1}_{Y_\lambda}(y) \, d\mu_\alpha(y) \\ &= C_{d,n} \int_{\mathbb{R}^{|n|}} \mathbf{1}_{Y_\lambda} \circ \phi(x) \, d\gamma(x) \\ &\le C_{d,n} \int_{\mathbb{R}^{|n|}} \mathbf{1}_{X_\lambda}(x) \, d\gamma(x) \\ &\le C_{d,n} \frac{C}{\lambda} \|f \circ \phi\|_{L^1(d\gamma)} \quad = \quad \frac{C}{\lambda} \|f\|_{L^1(d\mu_\alpha)}. \end{aligned}$$

□

REMARK 3.5.3. Riesz-Hermite transforms, unlike classical Riesz operators, are not of weak type 1-1 when the order is greater than 2, see [**FoSc, GMST2**]. Unfortunately, this negative result cannot be transferred to the Laguerre setting. There are two essential reasons for that: the lack of equality in Theorem 3.4.6 and the fact that the quadratic mapping ϕ is not injective.

THEOREM 3.5.4. *Let α be half-integer and assume that* $\min_i \alpha_i > -1/2$.

(a) *Given $p \in (1, \infty)$, the operators $K^\alpha_{m,j}$, $m \in \mathbb{N}^d$, $\bar{0} \le j \le m/2$, extend uniquely to bounded linear operators in $L^p(\mathbb{R}^d_+, d\mu_\alpha)$. The corresponding L^p constants depend neither on the dimension d nor on the type multi-index α.*
(b) *If $|m| \le 2$ then $K^\alpha_{m,j}$, $\bar{0} \le j \le m/2$, extend uniquely to weakly bounded linear operators on $L^1(\mathbb{R}^d_+, d\mu_\alpha)$.*

PROOF. Argue as in the proofs of Theorem 3.5.1 and 3.5.2, with the aid of Theorem 3.4.8. □

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