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Use of Distributions in Some Transforms in Optics

The applications of the mathematical theory of distributions for the solution of selected problems of wave optics are discussed in this paper. The generalized solution of the propagation of the mutual coherence is given.

1. Introduction

The theory of distributions is a mathematical tool for the solution of a variety of important problems in wave optics. It facilitates the exact formulation as well as the solution of many problems in e.g. optical imaging, holography (particularly in Fourier optics), diffraction etc., which is not possible within the range of a classical mathematical analysis. The results obtained by means of the distribution theory are more general in comparison with those gained by the classical analysis.

Possibilities of the use of the theory of distributions for the solution of certain problems in optical imaging are discussed in this article. The advantages of this theory are indicated. The concluding part of the article includes the formulae for the propagation of the mutual coherence function by means of distributions, which is a very important factor in the theory of imaging.

2. Differential Equations of Wave Optics

Optical phenomena of the wave nature can be mathematically expressed in the form of partial differential equations. Problems formulated on the basis of Maxwell equations of the electromagnetic field lead to the solution of the wave equation of the Helmholz equation for a monochromatic wave appropriate initial or boundary conditions.

By means of these equation more complex equations or their sets are set up, as e.g. the equations for the propagation of the mutual coherence function in isotropic or anisotropic media. The existence of the solution of these equations in the sense of a classical mathematical analysis imposes requirements on the derivatives of the functions, on the absolute term and on the initial and boundary conditions as regards their smoothness, which are fulfilled in practice only for a narrow range of problems. Strong smoothness requirements make a deeper analysis of the problem and a formation of precise conception about the behaviour of the system often impossible. On the other hand, if the required conditions are not respected, there is a risk of a faulty result.

The Kovalevskaya lemma concerning the solution of the Cauchy problem for the wave equation requires the functions mentioned above to be analytical, and guarantees the existence of the solution in a certain vicinity of the analytical point only [1].

The solution of the wave equation with constant coefficients at the second derivatives, having continuous derivatives up to the second order exists in a certain domain under the assumptions that the function of initial values has continuous derivatives up to the third order and that the time derivative has the derivatives up to the second order. The generalized solution of the wave equation not requiring this strong smoothness will be considered later.

3. Generalized Solutions

Generalized solutions may be defined in several ways. We shall consider generalized solution according to Sobolev [2] and the generalized solution expressed by distributions only.

3.1. Sobolev's Generalized Solution

Following [2], the generalized solution of a given differential equation in a certain domain is a function

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which is the limit of a sequence of ordinary solutions in the sense of considered typology. The generalized solution introduced in this way can even be a discontinuous function. It is sometimes, advantageous to seek generalized solutions in the space $L_2(G)$, which is a set of complex-valued functions of a real variable, for which the integral $\int_G |f|$ converges. It is a linear normalized space. One can define in it both: scalar product and the orthogonality of functions. $L_2(G)$ is a Hilbert space. It is convenient to use the Fourier series in this space.

It may be shown [1] that if the required smoothness of initial conditions is relinquished and if we suppose that the function of the initial value has continuous partial derivatives of the first order and continuous time derivative only, the solution of the wave equation with constant coefficients is a generalized solution of the Cauchy problem.

3.2. Distributions

The distribution is a linear continuous functional on a basic space. The basic space is the linear topological space of real-valued or complex-valued functions defined on a set of a real or complex n-dimensional point space. The topology of the basic space is generally supposed to be the countable normalized space.

The properties of distributions depend on the choice of the basic space. We shall, therefore, refer at first to several basic spaces.

3.2.1. Basic Spaces

***K(a)* -space** is a set of real-valued functions φ of n variables that have partial derivatives of all orders and equal zero outside the interval

$$[|x_1| \leqslant a_1, \ldots, |x_n| \leqslant a_n].$$

The topology is defined by a countable system of norms

$$\|\varphi\|_p = \sup_{|q| \leqslant p} |D^q \varphi(x)|, \quad p = 0, 1, \ldots,$$

The convergence corresponding to all norms is the uniform covergence of a sequence of functions and of all sequences of their derivatives.

K_n-space is a set of real-valued functions of n variables with continuous derivatives of all orders. Every function is equal to zero outside the bounded interval. The convergence is defined by the uniform convergence of $\varphi_\nu \in K_n$ and of all their derivatives on a bounded interval common for all φ_ν; outside this interval all these functions equal zero.

S_n-space is a set of functions with continuous derivatives of all orders for which the relation $|x^k| |D^q \varphi(x)| < C_{kq}$ $k = (k_1, \ldots, k_n)$, $q = (q_1, \ldots, q_n)$ applies. The sequence $\varphi_\nu \in S_n$ converges to φ then, and only then, if the derivatives of arbitrary order of the sequence φ_ν in a bounded interval converge uniformly to the corresponding derivatives of the function.

***Z*-space** is a set of integer functions $\varphi(z)$ of a complex variable for which the relation

$$|z|^q |\varphi(z)| \leqslant C_q \exp a|\tau|,$$

$a =$ const., $z = t + i\tau$, $q = 0, 1 \ldots$ applies.

D_m-space is a set of functions of one independent variable having continuous derivatives up to the m-th order. The function $\varphi \in D_m$ equals zero outside a bounded interval. The sequence φ_ν converges in D_m to the function φ then, and only then, if φ_ν equal zero outside the common interval and all sequences of derivatives up to the m-th order converge uniformly to the corresponding derivative of the function φ.

Basic spaces may be used for the solution of practical problems in wave optica. Distributions defined on basic spaces have some common properties, others are different. For example, the δ-distiibution can be defined on all above mentioned spaces. $\delta \in Z'$ is an analytical distribution and the relation

$$\delta(z+h) = \sum_{q=0}^{\infty} \delta^{(q)}(z) h/q!$$

is valid, where z, h are complex numbers. On the contrary, δ is in general not an analytical distribution if $\delta \in K_n'$, and $\delta^{(m+1)}$ is not defined even for $\delta \in D_m'$.

3.3. Solutions by Distributions

The solution of the differential equation, $P(D)f = 0$, where

$$P(D) = \sum_q a_q(x) D^q,$$

$$D^q = \frac{\partial^{|q|}}{\partial x_1^{q_1}, \ldots, \partial x_n^{q_1, q_n}},$$

$$q = (q_1, \ldots, q_n),$$

$$x = (x_1, \ldots, x_n),$$

in the sense of distributions, is a distribution f for which the relation

$$\big(f, P^*(D)\varphi\big) = 0,$$

where

$$P^*(D) = \sum_q (-1)^{|q|} \frac{\partial^{|q|}(a_q \varphi)}{\partial x_1^{q_1}, \ldots, \partial x_n^{q_n}}$$

is valid, φ is a function from one of the basic spaces. It is well known that the most advantageous method for finding the solution of differential equations in the form of distributions is the Fourier transform method. We often choose, therefore, the S space as a basic one in practice, for if $f \in S'$, the relation $F\{f\} \in S'$ is valid. The relations $F\{S\} = S$, $F\{S'\} = S'$ hold very often there [4]. This is not generally the case for other spaces, e.g. for $f \in K'$, $F\{f\} \in K'$ is not always valid, but $K \subset S$ and $S' \subset K'$ are.

4. Formulation of the Problem in the Sense of Distributions

4.1. Solution with the Continuous Dependence of the Variable Taken as a Parameter

The solution of the differential equation $P(D)f = 0$ as a distribution in K' exists always. The solution of this equation is $f = E * u$, where E is a fundamental function of the operator P and u is an absolute member or the distribution of the initial values. The solution of the equation can also be defined in the sense of distributions (generalized solution) with continuous dependence on the parameter which is one on the variables, the time variable t in our case.

For $t > 0$ we have

$$\left[\frac{\partial^n}{\partial t^n} + P_1(D)\right] f(t, x) =$$

$$\frac{\partial^n}{\partial t^n}[E(t, x) * u(x)] + P_1(D)[E(t, x) u(x)] =$$

$$\left\{\left[\frac{\partial^n}{\partial t^n} + P_1(D)\right] E(t, x)\right\} * u(x) = \Theta * u(x) = \Theta.$$

The generalized solution of the Cauchy problem for the wave equation is often defined in such a way. As an example we can give

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$$

with initial conditions $f(0, x) = 0$,

$$f_t(0, x) = u_0(x).$$

As $E(t, x) = \frac{1}{2}$ for $|x| < t$, $E(t, x) = 0$ for $|x| > t$, for $t > 0$ we have

$$\frac{\partial E}{\partial t} = \frac{1}{2}[\delta(x+t) + \delta(x-t)],$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{2}[\delta'(x+t) - \delta'(x-t)],$$

$$\frac{\partial E}{\partial x} = \frac{1}{2}[\delta(x+t) - \delta(x-t)],$$

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{2}[\delta'(x+t) - \delta'(x-t)].$$

Because of $f = E * u_0$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f = 0.$$

4.1.1. Dirichlet Problem

It is possible to define a problem similar to the Dirichlet problem of the classical analysis [3] even in the case of generalized solutions. Let E be the fundamental solution of the equation $P(D)u = 0$, U is the vicinity of the origin, $\alpha(x)$ is the function having all derivatives, $\alpha = 0$ outside U, $\alpha = 1$ on the set lying in U, W is a domain such that $W - U \subset V \subset G$, V is an internal domain in G where we seek the solution of the equation. $P(D)\,[(1-\alpha)E] * u(x) = u(x)$ applies then.

4.2. Solution by Distributions

In the sense of distributions, the solution of the differential equation $P(D)f_1 = f_2$ with boundary conditions $\Sigma b_m(x) f_1(x)$ given on a certain hypersurface T (the non-zero solution on one side of the hypersurface is mostly sought for) goes over to a task to find a distribution g_1 satisfying, in the sense of the generalized solution, the equation $P(D)g_1 = g_2 + g_3$, where g_1 or g_2 are distributions that equal f_1 or f_2 on one side of the hypersurface and equal zero on the other side of hypersurface. g_3 equals the contribution of the derivatives g_1 on T.

For the differential equation

$$\frac{\partial^M f_1}{\partial t^M} + \sum_{n=1}^{N} a_n \partial^n f_1 = f_2,$$

where

$$\sum_{n=1}^{N} a_n \partial^n = \sum_{n_1} \dots \sum_{n_m} a_{n_1}, \dots, a_{n_m} \frac{\partial^{n_1}}{\partial x_1^{n_1}}, \dots, \frac{\partial^{n_m}}{\partial x_m^{n_m}}$$

with initial conditions for $t > 0$

$$\frac{\partial^m f_1}{\partial t^m} = f_{1_m}(x_1, \dots, x_n),$$

$$m = 0, 1 \dots M-1$$

with

$$g_1(t, x) = f_1(x) H(t),$$

$$g_2(t, x) = f_2(x) H(t),$$

we get

$$\frac{\partial^M}{\partial t^M} g_1 + \sum_{n=1}^{N} a_n \partial^n_{g_1} = g_2 + g_3,$$

where

$$g_3 = \sum_{m=0}^{M-1} f_{1m} \delta^{(M-m-1)}(t).$$

If we know the fundamental solution E of the operator $P(D)$, the solution of $P(D)g = h$ is $g = E*h$.

Now we shall state some fundamental solutions.

4.2.1. Fundamental Solutions

The fundamental solution of the Helmholz equation in the n-dimensional space is

$$E_n^H = \frac{ik^{\frac{n}{2}-1} r^{1-\frac{n}{2}}}{2^{\frac{n}{2}+1} \pi^{\frac{n}{2}-1}} H^{(2)}_{\frac{n}{2}-1}(kr).$$

The fundamental solution of the wave equation is

$$E_1^W = \tfrac{1}{2} H(t) \cdot H(x^2 - t^2), \qquad n = 1,$$

$$E_2^W = \frac{H(t^2 - x_1^2 - x_2^2)}{2\pi \sqrt{(t^2 - x_1^2 - x_2^2)}}, \qquad n = 2,$$

$$E_3^W = 1/2\pi H(t) \delta(t^2 - x_1^2 - x_2^2 - x_3^2), \qquad n = 3.$$

4.2.2. Generalized Solutions

We now state the solution of the wave equation in free space. We make use of this solution for finding the propagation of the mutual coherence function, generally, in the non-stationary case.

The generalized solution of the wave equation in free space in the sense of distributions is

$$G_1 = \tfrac{1}{2} \int_S f_2 dS + f_1(x+t) + f_1(x-t), \qquad \text{for} \quad n = 1,$$

$$G_2 = \frac{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1{}^2} - \frac{\partial^2}{\partial x_2{}^2}\right)}{4\pi} \int_S \left\{ f_2 u_2(x-u) - f_1 \frac{\partial}{\partial u} u_2(x-u) \right\}_{u_1=0} dS, \qquad \text{for} \quad n = 2,$$

$$G_3 = \frac{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)}{4\pi} H(t) \times \times \left[\int_S f_2 dS + \int_\Omega f_1 d\Omega,\right] \qquad \text{for} \quad n = 3,$$

where f_1 and f_2 are functions of initial values and its derivative, respectively, S is the cone section $(t-\tau)^2 \geqslant (x_1 - u_1)^2 + \dots (x_n - u_n)^2$, $\tau \leqslant t$ formed by the hyperplane $t = 0$, Ω is the boundary of this section.

Using the preceding formula we obtain the solution of the equations for the propagation of the mutual coherence Γ in the non-stationary case for $n = 1$

$$\Gamma(Q_1, Q_2, t_1, t_2) = \tfrac{1}{4} \iint\limits_{\substack{t_1^2 \geqslant (x_1 - u_1)^2 \\ t_2^2 \geqslant (y_1 - v_1)^2}} \frac{\partial^2}{\partial t_1 \partial t_2} \times$$

$$\times \Gamma(P_1, P_2, 0, 0) dS_1 dS_2 + \tfrac{1}{2} \int\limits_{t_1^2 \geqslant (x_1 - u_1)^2} \left[\frac{\partial}{\partial t_1} \times \times \Gamma(P_1, y_1 - t_2, 0, 0) + \frac{\partial}{\partial t_1} \Gamma(P_1, y_1 + t_2, 0, 0)\right] dS +$$

$$+ \tfrac{1}{2} \int\limits_{t_2^2 \geqslant (y_1 - v_1)^2} \left[\frac{\partial}{\partial t_2} \Gamma(x_1 - t_1, P_2, 0, 0) + \frac{\partial}{\partial t_2} \times \times \Gamma(x_1 + t_1, P_2, 0, 0)\right] dS + \Gamma(x_1 - t_1, y_1 - t_2, 0, 0) +$$

$$+ \Gamma(x_1 - t_1, y_1 + t_2, 0, 0) + \Gamma(x_1 + t_1, y_1 - t_2, 0, 0) + \Gamma(x_1 + t_1, y_1 + t_2, 0, 0).$$

5. Conclusion

The conditions of the existence of the solution of partial differential equations which are the mathematical formulation of a certain physical problem, are often quite limited in practice and make an exact analysis of the problem impossible. Generalized solutions are, therefore, introduced in which these limiting conditions do not occur.

Generalized solutions may be defined in several ways. The definition of the generalized solution as a distribution is particularly advantageous in optics because it makes an objective geometrical interpretation of the propagation of the light disturbances.

As an example, the solution of the Helmholz equation in the form of distributions, where the right-hand side is the distribution δ, is the distribution $[\exp(-ikr)]/ /r$; in the case of the wave equation it is the distribution $[\delta(r-(t-t_0))]/r$. In the first case, we obtain a quasi-monochromatic wave, in the second — the propagation of a point disturbance from the source at $r = 0$ at the time $t = t_0$. This geometrical interpretation seems to be very advantageous in the theory of imaging, holography and diffraction phenomena.

References

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