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# Tolerance on Defect of Focus in the Presence of Spherical Aberration

The permissible defect of focus has been investigated in systems with spherical aberration. The simple analytical expressions giving the neighbourhood of the best imaging plane are determined using the conditions of perfect imaging of an incoherent by radiated point and spatial frequency, respectively.

## 1. Tolerance on defect of focus by spatial frequency imaging

The Strehl intensity ratio (Definitions-helligkeit) for a given focal plane is defined as the ratio of the maximum intensity $I$ (in the diffraction image in that plane) to the intensity $I_0$ obtained in the focal plane in the absence of aberration. The usually accepted value of Strehl intensity which is equivalent to the Rayleigh criterion is

$$\frac{I}{I_0} \geqslant 0.8. \tag{1}$$

In presence of spherical aberration the intensity on the optical axis in systems with circular aperture is given by the relation [1], [2]

$$I = I_0(C^2+S^2), \tag{2}$$

where

$$C = \int_0^1 \cos kW(q)\,dq \tag{3}$$

and

$$S = \int_0^1 \sin kW(q)\,dq. \tag{4}$$

Here $q$ means the square of the ratio of the height $h$ belonging to the incident ray to the height $h_k$ corresponding to the edge of the exit pupil [6]. The argument of the expressions (3) and (4) for the systems with the third, fifth and seventh-orders of spherical aberration, may be written [6]

$$kW(q) = \xi_{m,k}(A_1 q + A_3 q^2 + A_5 q^3 + A_7 q^4), \tag{5}$$

where $A_3, A_5, A_7$ are the coefficients of the third, fifth or seventh-order spherical aberration, respectively, and $A_1$ defines the position of the imaging plane. The coefficient $\xi_{m,k}$ depends on the spherical aberration $\Delta x_k$ corresponding to the edge of the exit pupil, or on the maximum spherical aberration $\Delta x_m$ occuring in the corrected systems [6].

If that argument is small, the sine and cosine functions may be expanded in a power series and expressed approximately by the first terms. Then (3) and (4) take the forms

$$C = \int_0^1 dq - \frac{1}{2}\xi_{m,k}^2 \int_0^1 [A_1 q + V(q)]^2 dq \tag{6}$$

and

$$S = \xi_{m,k}\int_0^1 [A_1 q + V(q)]\,dq, \tag{7}$$

where

$$V(q) = A_3 q^2 + A_5 q^3 + A_7 q^4. \tag{8}$$

Hence, the expression (2) may be written in the form

$$\frac{I}{I_0} = 1 + \xi_{m,k}^2 \Bigg\{\Big[\int_0^1 V(q)\,dq\Big]^2 - \int_0^1 V^2(q\,dq + + A_1\Big[\int_0^1 V(q)\,dq - 2\int_0^1 V(q)q\,dq\Big] - \frac{A_1^2}{12}\Bigg\}. \tag{9}$$

From the condition of the maximum intensity on the optical axis

$$\frac{dI}{dA_1} = 0 \tag{10}$$

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we find

$$\frac{\bar{A}_1}{6} = \int_0^1 V(q)dq - 2\int_0^1 V(q)q\,dq, \qquad (11)$$

where $\bar{A}_1$ characterizes the best position of the imaging plane.

After integration we obtain [6]

$$\bar{A}_1 = -\frac{1}{10}a, \qquad (12)$$

if we put

$$a = 10A_3 + 9A_5 + 8A_7. \qquad (13)$$

Let $x_0$ denote the distance of the imaging plane from the paraxial plane for the given monochromatic light of the wavelength $\lambda$ [7]. Then the best position of the imaging plane is

$$\bar{x}_0 = \frac{2}{5}\Delta x_{m,k}a, \qquad (14)$$

where $\Delta x_{m,k}$ is the common symbol of the maximum and edge spherical aberration. Using the relative defect of focus [6] we get

$$\zeta_{m,k} = \frac{2}{5}a. \qquad (15)$$

If we denote

$$b = \Big[\int_0^1 V(q)dq\Big]^2 - \int_0^1 V^2(q)dq$$
$$= -\Big(\frac{4}{45}A_3^2 + \frac{9}{112}A_5^2 + \frac{16}{225}A_7^2 +$$
$$+\frac{1}{6}A_3A_5 + \frac{16}{105}A_3A_5 + \frac{3}{20}A_5A_7\Big) \qquad (16)$$

and because

$$\int_0^1 V(q)dq - 2\int_0^1 V(q)q\,dq = -\frac{a}{60} \qquad (17)$$

the expression (9) may be written in form

$$\frac{I}{I_0} = 1 + \xi_{m,k}^2\Big(b - A_1\frac{a}{60} - A^2\frac{A_1^2}{12}\Big). \qquad (18)$$

For $\bar{A}_1 = A_1$ the condition (1) is given by the relation

$$-\xi_{m,k}^2\Big(\frac{a^2}{1200} + b\Big) \leq 0.2 \qquad (19)$$

and considering

$$\frac{a^2}{1200} + b < 0 \qquad (20)$$

we obtain

$$\xi_{m,k}^2 \leq -\frac{0.2}{\frac{a^2}{1200} + b}. \qquad (21)$$

Let $^{+}\Delta_{m,k}$ be the tolerance of the permissible spherical aberration $\Delta x_k$ and $\Delta x_m$ and $c$ the $f$-number of the optical system. Then the characteristic spherical aberration $\Delta x_{m,k}$ must fulfil the condition

$$|\Delta x_{m,k}| \leq |^{+}\Delta x_{m,k}| = \frac{\lambda c^2}{\pi}\sqrt{-\frac{0.2}{\frac{a^2}{1200} + b}}. \qquad (22)$$

Using the substitution of the relations (13) and (16) into (22) we get [6], [8]

$$^{+}\Delta x_{m,k} = \frac{12\lambda c^2}{\sqrt{35A_3^2 + 81A_5^2 + 112A_7^2 + 105A_3A_5 + 120A_3A_7 + 189A_5A_7}}. \qquad (23)$$

For the sake of simplicity the symbol of absolute value is omitted.

The coefficients $A_3, A_5, A_7$ are functions of the height ratio $h_0/h_k$ [6]. Then the tolerance $^{+}\Delta x_{m,k}$ can be also expressed as the function of $h_0/h_k$. Assuming $^{+}\bar{x}_0$ to be the best imaging plane corresponding to the tolerance $^{+}\Delta x_{m,k}$ the latter can be also presented as the function of the correction $h_0/h_k$. The values $^{+}\Delta x_{m,k}$ and $^{+}\bar{x}_0$, as well as the relative defect of focus $\zeta_{m,k}$ belonging to the position of the best imaging plane for the fifth-order spherical aberration are given in Table 1. It should be noted that Table 1 includes only absolute values of $^{+}\Delta x_{m,k}$. Considering the orientation, this value has to be determined according to the relation $\Delta x_{m,k} = \zeta_{m,k}x_0$. The tolerance $^{+}\Delta x_k$ of seventh-order spherical aberration as well as the relative defect of focus $\zeta_k$ can be found in [6].

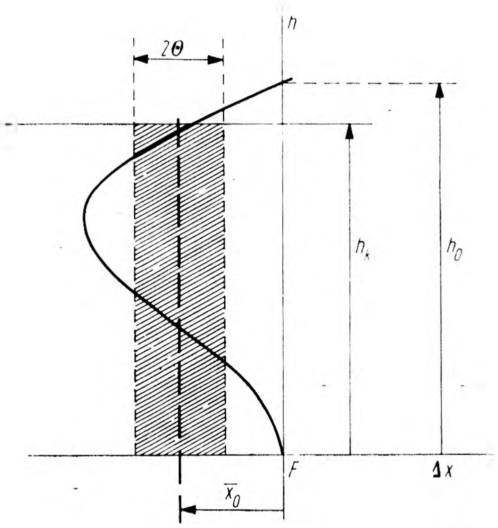
Assuming that $\Delta x_{m,k}$ is less than the tolerance $^{+}\Delta x_{m,k}$ we expect that a permissible interval of the defect of focus will not exceed $2\theta$ in the neighbourhood of the best imaging plane $\bar{x}_0$ (Fig. ). This defect may be determined from the relation (18). The condition of perfect imaging may be written in the form

$$-\xi_{m,k}^2(b - A_1\frac{a}{60} - \frac{A_1^2}{12} \leq 0.2 \qquad (24)$$

Table 1

Tolerances $^{+}\Delta x_m$ and $^{+}\Delta x_k$ of the fifth-order spherical aberration in multiples of $\lambda c^2$, the corresponding values of the best imaging plane $^{+}\bar{x}_0$, and the best relative defect of focus $\bar{\zeta}_m$ and $\bar{\zeta}_k$ in versus the height correction $h_0/h_k$

| $h_0/h_k$ | $^{+}\Delta x_m$ | $^{+}\Delta x_k$ | $\bar{\zeta}_m$ | $\bar{\zeta}_k$ | $^{+}\bar{x}_0$ |
|---|---|---|---|---|---|
| 0.20 | 0.007 | 16.00 | −700.0 | 0.292 | −4.66 |
| 0.40 | 0.121 | 15.91 | − 34.4 | −0.262 | −4.17 |
| 0.50 | 0.33 | 15.83 | − 11.2 | 0.233 | −3.69 |
| 0.60 | 0.79 | 15.70 | − 3.7 | 0.187 | −2.94 |
| 0.65 | 1.20 | 15.60 | − 1.99 | 0.154 | −2.39 |
| 0.70 | 1.81 | 15.40 | − 0.916 | 0.108 | −1.66 |
| 0.75 | 2.74 | 15.14 | − 0.237 | 0.043 | −0.65 |
| 0.80 | 4.18 | 14.70 | 0.195 | −0.055 | 0.82 |
| 0.85 | 6.52 | 13.90 | 0.469 | −0.221 | 3.06 |
| 0.90 | 10.47 | 12.12 | 0.640 | −0.553 | 6.70 |
| 0.95 | 16.93 | 8.11 | 0.743 | −1.551 | 12.58 |
| 1.00 | 24.00 | 0.00 | 0.800 | − − − | 19.2 |
| 1.05 | 24.94 | 8.41 | 0.827 | 2.451 | 20.62 |
| 1.10 | 22.03 | 12.64 | 0.833 | 1.452 | 18.36 |
| 1.15 | 19.49 | 14.37 | 0.826 | 1.120 | 16.10 |
| 1.20 | 17.85 | 15.15 | 0.810 | 0.954 | 14.46 |
| 1.25 | 16.86 | 15.54 | 0.788 | 0.855 | 13.29 |
| 1.30 | 16.31 | 15.76 | 0.763 | 0.790 | 12.45 |




The permissible interval of defect of focus in the neighbourhood of the best imaging plane $\bar{x}_0$

or

$$\left(A_1+\frac{a}{10}\right)^2-B^2 \leqq 0, \tag{25}$$

where

$$B^2 = 12\left(\frac{0.2}{\xi^2_{m,k}}+\frac{a^2}{1200}+b\right). \tag{26}$$

By virtue of (22) this relation can be modified to the form

$$B^2 = 2.4\left(\frac{1}{\xi^2_{m,k}}-\frac{1}{^{+}\xi^2_{m,k}}\right). \tag{27}$$

Hence

$$\xi^2_{m,k} \leqq {}^{+}\xi^2_{m,k}.$$

The last ineqality is identically fulfilled when

$$\left|A_1+\frac{a}{10}\right| \leqq B \tag{28}$$

provided that $B$ is nonnegative. Then we can write

$$|\bar{\zeta}-\zeta| \leqq 4B \tag{29}$$

or

$$|x_0-\bar{x}_0| \leqq \theta, \tag{30}$$

where

$$\theta = 4B\,|\Delta x_{m,k}|. \tag{31}$$

The modification of (31) by means of (27) leads finally to

$$\theta \doteq 2\lambda c^2\sqrt{1-\left(\frac{\Delta x_{m,k}}{^{+}\Delta x_{m,k}}\right)^2} \tag{32}$$

if

$$\frac{4\sqrt{2.4}}{\pi} \doteq 2.$$

The tolerances $^{+}\Delta x_{m,k}$ related to the ratio $h_0/h_k$ can be explicitly given from the relation (23) or by a suitable approximation determined from the Table 1. The tolerances $^{+}\Delta x_k$, corresponding to the seventh-order spherical aberration, were approximately evaluated in [6]. Hence, for $\Delta x_{m,k} < {}^{+}\Delta x_{m,k}$ the permissible defect $0 \leqq \theta \leqq 2\lambda c^2$ can be always determined. In systems without spherical aberration ($\Delta x_{m,k} = 0$) the interval $2\theta = 4\lambda c^2$, whereas in the case of $\Delta x_{m,k} = {}^{+}\Delta x_{m,k}$ the interval $2\theta = 0$. Subsequently, for $\theta = 0$ there is only one imaging plane fulfilling the condition of the perfect imaging of a point.

## 2. Tolerance on defect of focus by spatial frequency imaging

Studying the best correction of the spherical aberration in dependence on the spatial frequency $R$ we can start with the approximation of the response function in the form [7]

$$D(\varrho) = D_0(\varrho)-D_1(\varrho), \tag{33}$$

where $\varrho = R\lambda c$ is the normalized spatial frequency, $D_0(\varrho)$ is the response function of the ideal optical system, and $D_1(\varrho)$ the function depending on the correction of the system

[3], [7]. In the case of the spherical aberration, the aberration function has the form [7]

$$D_1(\varrho) = \left(\frac{\Delta x_{m,k}}{\lambda c^2}\right)^2 [A_1^2 T_1(\varrho) + A_3^2 T_2(\varrho) + A_5^2 T_3 + (\varrho) A_7^2 T_4(\varrho) + A_1 A_3 T_5(\varrho) + A_1 A_5 T_6(\varrho) + A_1 A_7 T_7(\varrho) + A_3 A_5 T_8(\varrho) + A_3 A_7 T_9(\varrho) + A_5 A_7 T_{10}(\varrho). \quad (34)$$

The functions $T_i(\varrho)$ can be either calculated explicitly from the relations (22) and (23) of [7] or determined approximatively using the tabulated values given in the same paper.

To determine the position of the best imaging plane depending on the spatial frequency we start with the condition of the maximum contrast on the optical axis

$$\frac{\partial D(\varrho)}{\partial A_1} = 0. \quad (35)$$

The best position of the imaging plane is then expressed by

$$\bar{A}_1(\varrho) = -\frac{1}{2T_1(\varrho)} [A_3 T_5(\varrho) + A_5 T_6(\varrho) + A_7 T_7(\varrho)] \quad (36)$$

or better by

$$\zeta(\varrho) = \frac{2}{T_1(\varrho)} a(\varrho)$$

and

$$\bar{x}_0(\varrho) = \frac{2}{T_1(\varrho)} \Delta x_{m,k}\, a(\varrho), \quad (37)$$

where, by analogy

$$a(\varrho) = A_3 T_5(\varrho) + A_5 T_6(\varrho) + A_7 T_7(\varrho). \quad (38)$$

In view of (38) we can write (33) in the form

$$D_1(\varrho) = \left(\frac{\Delta x_{m,k}}{\lambda c^2}\right)^2 [A_1^2 T_1(\varrho) + A_1 a(\varrho) + b(\varrho)] \quad (39)$$

considering

$$b(\varrho) = A_3^2 T_2(\varrho) + A_5^2 T_3(\varrho) + A_7^2 T_4(\varrho) + A_3 A_5 T_8(\varrho) + A_3 A_7 T_9(\varrho) + A_5 A_7 T_{10}(\varrho). \quad (40)$$

The condition of perfect imaging of the spatial frequency [3]

$$\frac{D(\varrho)}{D_0(\varrho)} \geqq 0.8 \quad (41)$$

implies

$$\left(\frac{\Delta x_{m,k}}{\lambda c^2}\right)^2 [A_1^2 T_1(\varrho) + A_1 a(\varrho) + b(\varrho)] \leqq 0.2 D_0(\varrho) \quad (42)$$

and

$$\left[A_1 + \frac{a(\varrho)}{2T_1(\varrho)}\right]^2 - B^2(\varrho) \leqq 0 \quad (43)$$

taking the substitution

$$B^2(\varrho) = \frac{a^2(\varrho)}{4T_1^2(\varrho)} + \frac{0.2 D_0(\varrho)}{T_1(\varrho)\left(\frac{\Delta x_{m,k}}{\lambda c^2}\right)^2} - \frac{b(\varrho)}{T_1(\varrho)} \quad (44)$$

and with respect to $T_1(\varrho) \geqq 0$. Taking no account of the sign at $\Delta x_{m,k}$ we get from (42) using $A_1 = \bar{A}_1(\varrho)$

$$\Delta x_{m,k} = {}^{+}\Delta x_{m,k}(\varrho) = \sqrt{\frac{0.8 D_0(\varrho) T_1(\varrho)}{4T_1(\varrho) b(\varrho) - a^2(\varrho)}}\, \lambda c^2. \quad (45)$$

Here, ${}^{+}\Delta x_{m,k}(\varrho)$ is the maximum of the permissible spherical aberration enabling the perfect imaging of the spatial frequency $\varrho$. The final form of the relation (45) referring to the correction coefficients $A_3$, $A_5$, $A_7$ is given in [7].

The tolerances ${}^{+}\Delta x_m(\varrho)$ and ${}^{+}\Delta x_k(\varrho)$ of the fifth-order spherical aberrations, in dependence on $h_0/h_k$, are given in Tables 2 and 3. The values ${}^{+}\bar{x}(\varrho)$ giving the best position of the imaging plane with respect to the tolerances ${}^{+}\Delta x_{m,k}(\varrho)$ are presented in Table 4. The quantities $\zeta_{m,k}(\varrho)$ corresponding to the best contrast on the optical axis are given in Tables 5 and 6. The optimum corrections and tolerances of the fifth-order spherical aberration are given in [9]. The optimum corrections and tolerances of the seventh-order spherical aberration have been studied and determined with respect to the best imaging of a spatial frequency in [7].

Because of $\Delta x_{m,k} < {}^{+}\Delta x_{m,k}(\varrho)$ an interval $2\theta(\varrho)$ of the permissible defect of focus should also occur in the neighbourhood of the best imaging plane $\bar{x}_0(\varrho)$. In view of the relation (43) and writing

$$B^2(\varrho) = \frac{0.2 D_0(\varrho)}{T_1(\varrho)} \left\{\frac{1}{\left(\frac{\Delta x_{m,k}}{\lambda c^2}\right)^2} - \frac{1}{\left[\frac{{}^{+}\Delta x_{m,k}(\varrho)}{\lambda c^2}\right]^2}\right\} \quad (46)$$

Table 2

Tolerances $^{+}\Delta x_m(\varrho)$ of the fifth-order spherical aberration in multiples of $\lambda c^2$ versus the height correction $h_0/h_k$
$^{+}\Delta x_m(\varrho)$

| $h_0/h_k$ / $\varrho$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | 0.6 | 1.4 | 3.5 | 7.6 | 16.1 | 35.0 | 78.8 | 137 | 128 | 111 | 104 | 102 |
| 0.04 | 0.3 | 0.8 | 1.9 | 4.2 | 9.1 | 20.1 | 45.6 | 71.5 | 63.6 | 55.9 | 53.0 | 52.6 |
| 0.06 | 0.2 | 0.6 | 1.4 | 3.1 | 6.8 | 15.3 | 34.9 | 49.1 | 42.4 | 37.8 | 36.2 | 36.2 |
| 0.08 | 0.2 | 0.5 | 1.2 | 2.6 | 5.7 | 13.1 | 29.7 | 37.8 | 32.0 | 28.9 | 27.9 | 28.1 |
| 0.10 | 0.2 | 0.4 | 1.0 | 2.3 | 5.1 | 12.0 | 26.6 | 30.8 | 26.0 | 23.7 | 23.0 | 23.3 |
| 0.12 | 0.1 | 0.4 | 0.9 | 2.1 | 4.8 | 11.3 | 24.6 | 26.2 | 22.1 | 20.3 | 19.9 | 20.3 |
| 0.14 | 0.1 | 0.4 | 0.9 | 2.0 | 4.6 | 11.0 | 23.1 | 23.0 | 19.5 | 18.0 | 17.8 | 18.2 |
| 0.16 | 0.1 | 0.3 | 0.8 | 1.9 | 4.4 | 10.8 | 22.1 | 20.8 | 17.6 | 16.4 | 16.3 | 16.7 |
| 0.18 | 0.1 | 0.3 | 0.8 | 1.9 | 4.3 | 10.7 | 21.5 | 19.2 | 16.3 | 15.3 | 15.2 | 15.6 |
| 0.20 | 0.1 | 0.3 | 0.8 | 1.8 | 4.3 | 10.8 | 20.7 | 18.0 | 15.3 | 14.4 | 14.3 | 14.8 |
| 0.25 | 0.1 | 0.3 | 0.8 | 1.8 | 4.2 | 11.0 | 20.2 | 16.5 | 14.0 | 13.2 | 13.2 | 13.6 |
| 0.30 | 0.1 | 0.3 | 0.7 | 1.7 | 4.2 | 11.3 | 20.9 | 16.3 | 13.7 | 12.9 | 12.9 | 13.3 |
| 0.35 | 0.1 | 0.3 | 0.7 | 1.7 | 4.2 | 11.3 | 22.4 | 17.0 | 14.1 | 13.1 | 13.1 | 13.6 |
| 0.40 | 0.1 | 0.3 | 0.7 | 1.7 | 4.1 | 11.0 | 23.8 | 18.5 | 15.0 | 13.9 | 13.8 | 14.1 |
| 0.45 | 0.1 | 0.3 | 0.8 | 1.7 | 4.1 | 10,5 | 24.4 | 20.8 | 16.7 | 15.2 | 14.9 | 15.2 |
| 0.50 | 0.1 | 0.3 | 0.8 | 1.8 | 4.0 | 9.9 | 24.1 | 24.0 | 19.1 | 17.2 | 16.7 | 16.9 |
| 0.55 | 0.1 | 0.3 | 0.8 | 1.8 | 4.1 | 9.6 | 23.4 | 28.5 | 22.7 | 20.1 | 19.3 | 19.4 |
| 0.60 | 0.1 | 0.4 | 0.9 | 1.9 | 4.2 | 9.5 | 23.0 | 34.7 | 28.4 | 24.5 | 23.1 | 23.0 |
| 0.65 | 0.2 | 0.4 | 0.9 | 2.1 | 4.4 | 9.8 | 23.0 | 42.8 | 37.7 | 31.6 | 29.1 | 28.5 |
| 0.70 | 0.2 | 0.5 | 1.1 | 2.3 | 4.9 | 10.4 | 23.5 | 51.1 | 53.8 | 43.7 | 39.0 | 37.4 |
| 0.75 | 0.2 | 0.6 | 1.3 | 2.8 | 5.7 | 11.8 | 25.8 | 61.2 | 86.6 | 67.6 | 57.2 | 53.2 |
| 0.80 | 0.5 | 0.7 | 1.7 | 3.5 | 6.9 | 13.7 | 27.9 | 59.7 | 110 | 108 | 91.6 | 83.1 |
| 0.85 | 0.4 | 1.0 | 2.1 | 4.2 | 8.1 | 14.9 | 26.8 | 47.2 | 79.2 | 114 | 133 | 137 |

Table 3

Tolerances $^{+}\Delta x_k(\varrho)$ of the fifth-order spherical aberration in multiples of $\lambda c^2$ versus the height correction $h_0/h_k$
$^{+}\Delta x_k(\varrho)$

| $h_0/h_k$ / $\varrho$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | 73.7 | 71.9 | 69.2 | 64.7 | 56.8 | 40.5 | 0.0 | 78.4 | 108 | 107 | 103 | 101 |
| 0.04 | 40.4 | 39.5 | 38.2 | 36.0 | 32.0 | 23.2 | 0.0 | 41.0 | 54.0 | 54.0 | 52.9 | 52.0 |
| 0.06 | 29.5 | 28.9 | 28.1 | 26.7 | 24.0 | 17.8 | 0.0 | 28.2 | 36.0 | 36.5 | 36.1 | 35.7 |
| 0.08 | 24.1 | 23.8 | 23.2 | 22.2 | 20.2 | 15.2 | 0.0 | 21.6 | 27.2 | 27.9 | 27.9 | 27.8 |
| 0.10 | 21.0 | 20.8 | 20.4 | 19.6 | 18.1 | 13.9 | 0.0 | 17.6 | 22.1 | 22.9 | 23.0 | 23.0 |
| 0.12 | 19.1 | 18.9 | 18.6 | 18.1 | 16.8 | 13.1 | 0.0 | 15.0 | 18.8 | 19.6 | 19.9 | 20.0 |
| 0.14 | 17.7 | 17.6 | 17.4 | 17.0 | 16.0 | 12.7 | 0.0 | 13.2 | 16.5 | 17.4 | 17.8 | 17.9 |
| 0.16 | 16.8 | 16.7 | 16.6 | 16.3 | 15.5 | 12.5 | 0.0 | 11.9 | 15.0 | 15.9 | 16.2 | 16.4 |
| 0.18 | 16.1 | 16.0 | 16.0 | 15.8 | 15.2 | 12.4 | 0.0 | 11.0 | 13.8 | 14.7 | 15.1 | 15.4 |
| 0.20 | 15.6 | 15.6 | 15.6 | 15.5 | 15.0 | 12.5 | 0.0 | 10.3 | 13.0 | 13.9 | 14.3 | 14,6 |
| 0.25 | 14.9 | 14.9 | 14.9 | 14.9 | 14.8 | 12.7 | 0.0 | 9.5 | 11.9 | 12.8 | 13.2 | 13.5 |
| 0.30 | 14.6 | 14.7 | 14.8 | 14.8 | 14.8 | 13.1 | 0.0 | 9.3 | 11.6 | 12.4 | 12.9 | 13.1 |
| 0.35 | 14.6 | 14.6 | 14.7 | 14.8 | 14.7 | 13.1 | 0.0 | 9.7 | 11.9 | 12.7 | 13.1 | 13.3 |
| 0.40 | 14.8 | 14.8 | 14.8 | 14.7 | 14.5 | 12.7 | 0.0 | 10.6 | 12.8 | 13.5 | 13.8 | 13.9 |
| 0.45 | 15.1 | 15.1 | 15.0 | 14.8 | 14.3 | 12.1 | 0.0 | 11.9 | 14.1 | 14.7 | 14.9 | 15.0 |
| 0.50 | 15.8 | 15.6 | 15.4 | 15.0 | 14.2 | 11.5 | 0.0 | 13.8 | 16.2 | 16.6 | 16.7 | 16.7 |
| 0.55 | 16.7 | 16.5 | 16.1 | 15.5 | 14.3 | 11.1 | 0.0 | 16.5 | 19.3 | 19.4 | 19.3 | 19.1 |
| 0.60 | 18.1 | 17.8 | 17.3 | 16.4 | 14.7 | 11.0 | 0.0 | 19.9 | 24.1 | 23.7 | 23.1 | 22.7 |
| 0.65 | 20.2 | 19.7 | 18.9 | 17.8 | 15.6 | 11.3 | 0.0 | 24.5 | 32.0 | 30.5 | 9.1 | 28.2 |
| 0.70 | 23.2 | 22.6 | 21.5 | 19.9 | 17.2 | 12.2 | 0.0 | 29.3 | 45.7 | 42.3 | 39.0 | 37.0 |
| 0.75 | 28.3 | 27.3 | 25.8 | 23.6 | 20.0 | 13.6 | 0.0 | 35.1 | 73.5 | 65.4 | 57.2 | 52.5 |
| 0.80 | 36.6 | 35.0 | 32.8 | 29.5 | 24.4 | 15.9 | 0.0 | 34.3 | 93.1 | 105 | 91.6 | 82.1 |
| 0.85 | 48.2 | 45.4 | 41.6 | 36.2 | 28.6 | 17.3 | 0.0 | 27.1 | 67.3 | 111 | 133 | 135 |

Table 4

The best imaging plane $^{+}\bar{x}_0(\varrho)$ in multiples of $\lambda c^2$ corresponding to the tolerances $^{+}\Delta x_{m,k}(\varrho)$ versus the height correction $h_0/h_k$

$^{+}\bar{x}_0(\varrho)$

| $h_0/h_k$ / $\varrho$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | −32.0 | −29.4 | −25.5 | −19.3 | −8.7 | 11.6 | 55.8 | 116 | 114 | 96.8 | 86.2 | 79.6 |
| 0.04 | −16.3 | −14.9 | −12.8 | −9.4 | −3.5 | 8.2 | 33.8 | 61.8 | 56.4 | 48.2 | 43.2 | 40.1 |
| 0.06 | −11.1 | −10.1 | −8.6 | −6.1 | −1.7 | 7.2 | 26.8 | 42.8 | 37.4 | 32.2 | 29.0 | 27.1 |
| 0.08 | −8.5 | −7.7 | −6.5 | −4.4 | −0.8 | 6.8 | 23.4 | 32.9 | 28.1 | 24.3 | 22.0 | 20.6 |
| 0.10 | −6.9 | −6.3 | −5.2 | −3.4 | −0.2 | 6.7 | 21.4 | 26.9 | 22.6 | 19.6 | 17.9 | 16.8 |
| 0.12 | −5.9 | −5.3 | −4.4 | −2.8 | 0.2 | 6.8 | 20.0 | 22.8 | 19.0 | 16.6 | 15.2 | 14.3 |
| 0.14 | −5.2 | −4.7 | −3.8 | −2.3 | 0.5 | 6.8 | 18.9 | 20.0 | 16.6 | 14.5 | 13.3 | 12.6 |
| 0.16 | −4.7 | −4.2 | −3.4 | −2.0 | 0.7 | 7.0 | 18.1 | 17.9 | 14.8 | 13.0 | 12.0 | 11.5 |
| 0.18 | −4.3 | −3.8 | −3.0 | −1.7 | 0.9 | 7.1 | 17.5 | 16.4 | 13.5 | 11.9 | 11.0 | 10.4 |
| 0.20 | −4.0 | −3.5 | −2.8 | −1.5 | 1.0 | 7.2 | 17.0 | 15.3 | 12.6 | 11.1 | 10.2 | 9.7 |
| 0.25 | −3.5 | −3.1 | −2.4 | −1.2 | 1.3 | 7.5 | 16.4 | 13.6 | 11.1 | 9.8 | 9.1 | 8.6 |
| 0.30 | −3.2 | −2.8 | −2.2 | −1.0 | 1.4 | 7.7 | 16.7 | 13.2 | 10.6 | 9.3 | 8.6 | 8.1 |
| 0.35 | −3.2 | −2.8 | −2.1 | −0.9 | 1.4 | 7.7 | 17.6 | 13.5 | 10.7 | 9.4 | 8.6 | 8.1 |
| 0.40 | −3.2 | −2.8 | −2.1 | −1.0 | 1.3 | 7.4 | 18.6 | 14.7 | 11.4 | 9.9 | 9.0 | 8.5 |
| 0.45 | −3.4 | −2.9 | −2.2 | −1.1 | 1.3 | 7.0 | 19.1 | 16.6 | 12.8 | 11.0 | 9.9 | 9.2 |
| 0.50 | −3.7 | −3.2 | −2.5 | −1.2 | 1.1 | 6.6 | 19.1 | 19.5 | 14.9 | 12.6 | 11.3 | 10.5 |
| 0.55 | −4.2 | −3.7 | −2.8 | −1.5 | 1.0 | 6.3 | 18.9 | 23.7 | 18.3 | 15.2 | 13.5 | 12.5 |
| 0.60 | −5.0 | −4.3 | −3.4 | −1.9 | 0.8 | 6.2 | 18.9 | 29.8 | 23.7 | 19.3 | 16.9 | 15.5 |
| 0.65 | −6.2 | −5.4 | −4.3 | −2.6 | 0.4 | 6.1 | 19.1 | 37.9 | 32.8 | 26.1 | 22.4 | 20.3 |
| 0.70 | −8.1 | −7.2 | −5.8 | −3.7 | −0.3 | 6.0 | 19.6 | 48.8 | 48.8 | 37.9 | 31.7 | 28.2 |
| 0.75 | −11.4 | −10.2 | −8.5 | −5.8 | −1.6 | 5.8 | 21.0 | 56.6 | 81.3 | 61.4 | 49.0 | 42.4 |
| 0.80 | −17.4 | −15.7 | −13.2 | −9.7 | −4.2 | 4.8 | 21.2 | 55.1 | 106 | 102 | 82.5 | 70.1 |
| 0.85 | −27.3 | −24.6 | −20.9 | −15.8 | −8.6 | 2.0 | 17.9 | 42.4 | 77.1 | 112 | 125 | 122 |

Table 5

The relative defect of focus $\bar{\zeta}_m(\varrho)$ characterizing the best imaging plane versus on $h_0/h_k$

$\bar{\zeta}_m(\varrho)$

| $h_0/h_k$ / $\varrho$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | −57.0 | −19.6 | −7.3 | −2.53 | −0.54 | 0.33 | 0.71 | 0.85 | 0.89 | 0.87 | 0.83 | 0.78 |
| 0.04 | −53 | −18 | −6.6 | −2.21 | −0.38 | 0.41 | 0.74 | 0.86 | 0.89 | 0.86 | 0.82 | 0.76 |
| 0.06 | −49 | −16.7 | −6.0 | −1.94 | −0.25 | 0.47 | 0.77 | 0.87 | 0.88 | 0.85 | 0.80 | 0.75 |
| 0.08 | −46 | 15.5 | −5.5 | −1.70 | −0.14 | 0.52 | 0.79 | 0.87 | 0.88 | 0.84 | 0.79 | 0.75 |
| 0.10 | −43 | −14.4 | −5.0 | −1.49 | −0.04 | 0.56 | 0.80 | 0.87 | 0.87 | 0.83 | 0.78 | 0.72 |
| 0.12 | −42 | −13.5 | −4.6 | −1.31 | 0.04 | 0.60 | 0.81 | 0.87 | 0.86 | 0.82 | 0.76 | 0.71 |
| 0.14 | −38 | −12.7 | −4.3 | −1.16 | 0.11 | 0.62 | 0.82 | 0.87 | 0.85 | 0.80 | 0.75 | 0.69 |
| 0.16 | −37 | −12.0 | −4.0 | −1.03 | 0.16 | 0.65 | 0.82 | 0.86 | 0.84 | 0.79 | 0.74 | 0.68 |
| 0.18 | −35 | −11.4 | −3.7 | −0.92 | 0.21 | 0.66 | 0.82 | 0.85 | 0.83 | 0.78 | 0.72 | 0.67 |
| 0.20 | −33 | −10.8 | −3.5 | −0.83 | 0.25 | 0.67 | 0.82 | 0.85 | 0.82 | 0.77 | 0.71 | 0.66 |
| 0.25 | −31 | −9.9 | −3.1 | −0.67 | 0.30 | 0.68 | 0.81 | 0.83 | 0.79 | 0.74 | 0.69 | 0.63 |
| 0.30 | −29 | −9.3 | −2.9 | −0.59 | 0.33 | 0.68 | 0.80 | 0.81 | 0.77 | 0.72 | 0.67 | 0.61 |
| 0.35 | −28 | −9.1 | −2.8 | −0.56 | 0.33 | 0.68 | 0.79 | 0.80 | 0.76 | 0.71 | 0.66 | 0.61 |
| 0.40 | −28 | −9.1 | −2.9 | −0.57 | 0.32 | 0.67 | 0.78 | 0.79 | 0.76 | 0.71 | 0.65 | 0.60 |
| 0.45 | −29 | −9.3 | −3.0 | −0.61 | 0.31 | 0.66 | 0.78 | 0.80 | 0.76 | 0.71 | 0.69 | 0.63 |
| 0.50 | −31 | −9.9 | −3.2 | −0.69 | 0.28 | 0.66 | 0.79 | 0.81 | 0.78 | 0.73 | 0.68 | 0.62 |
| 0.55 | −32 | −10.6 | −3.5 | −0.81 | 0.24 | 0.66 | 0.81 | 0.83 | 0.80 | 0.76 | 0.70 | 0.64 |
| 0.60 | −36 | −11.7 | −3.9 | −0.98 | 0.18 | 0.65 | 0.82 | 0.86 | 0.84 | 0.79 | 0.73 | 0.67 |
| 0.65 | −40 | −13.2 | −4.5 | −1.24 | 0.08 | 0.63 | 0.83 | 0.89 | 0.87 | 0.83 | 0.77 | 0.71 |
| 0.70 | −46 | −15.3 | −5.3 | −1.60 | −0.06 | 0.58 | 0.83 | 0.91 | 0.91 | 0.87 | 0.81 | 0.75 |
| 0.75 | −53 | −18.0 | −6.5 | −2.10 | −0.28 | 0.49 | 0.81 | 0.92 | 0.94 | 0.91 | 0.86 | 0.80 |
| 0.80 | −62 | −21.5 | −8.0 | −2.79 | −0.61 | 0.35 | 0.76 | 0.92 | 0.96 | 0.94 | 0.90 | 0.84 |
| 0.85 | −74 | −26.0 | −9.9 | −3.71 | −1.05 | 0.13 | 0.67 | 0.90 | 0.97 | 0.97 | 0.94 | 0.89 |

Table 6

The relative defect of focus $\bar{\zeta}_k(\varrho)$ characterizing the best imaging plane versus on $h_0/h_k$

$\bar{\zeta}_k(\varrho)$

| $h_0/h_k$ / $\varrho$ | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.02 | 0.43 | 0.41 | 0.37 | 0.30 | 0.15 | −0.29 | $-10^7$ | 1.49 | 1.05 | 0.90 | 0.83 | 0.79 |
| 0.04 | 0.40 | 0.38 | 0.33 | 0.26 | 0.11 | −0.35 | $-10^7$ | 1.51 | 1.04 | 0.89 | 0.82 | 0.77 |
| 0.06 | 0.38 | 0.39 | 0.30 | 0.23 | 0.07 | −0.40 | $-10^7$ | 1.52 | 1.04 | 0.88 | 0.80 | 0.76 |
| 0.08 | 0.35 | 0.32 | 0.28 | 0.20 | 0.04 | −0.45 | $-10^7$ | 1.52 | 1.03 | 0.87 | 0.79 | 0.74 |
| 0.10 | 0.33 | 0.30 | 0.25 | 0.17 | 0.01 | −0.49 | $-10^7$ | 1.52 | 1.02 | 0.86 | 0.78 | 0.73 |
| 0.12 | 0.31 | 0.28 | 0.23 | 0.15 | −0.01 | −0.52 | $-10^7$ | 1.52 | 1.01 | 0.85 | 0.76 | 0.71 |
| 0.14 | 0.29 | 0.26 | 0.22 | 0.14 | −0.03 | −0.54 | $-10^7$ | 1.51 | 1.00 | 0.83 | 0.75 | 0.70 |
| 0.16 | 0.28 | 0.25 | 0.20 | 0.12 | −0.05 | −0.56 | $-10^7$ | 1.50 | 0.99 | 0.82 | 0.74 | 0.69 |
| 0.18 | 0.27 | 0.24 | 0.19 | 0.11 | −0.06 | −0.57 | $-10^7$ | 1.49 | 0.98 | 0.81 | 0.72 | 0.67 |
| 0.20 | 0.26 | 0.23 | 0.18 | 0.10 | −0.07 | −0.58 | $-10^7$ | 1.48 | 0.97 | 0.80 | 0.71 | 0.66 |
| 0.25 | 0.23 | 0.21 | 0.16 | 0.08 | −0.09 | −0.59 | $-10^7$ | 1.44 | 0.94 | 0.77 | 0.69 | 0.64 |
| 0.30 | 0.22 | 0.19 | 0.15 | 0.07 | −0.09 | −0.59 | $-10^7$ | 1.41 | 0.91 | 0.75 | 0.67 | 0.62 |
| 0.35 | 0.22 | 0.19 | 0.14 | 0.07 | −0.10 | −0.58 | $-10^7$ | 1.39 | 0.90 | 0.74 | 0.66 | 0.61 |
| 0.40 | 0.22 | 0.19 | 0.14 | 0.07 | −0.09 | −0.58 | $-10^7$ | 1.38 | 0.89 | 0.73 | 0.65 | 0.61 |
| 0.45 | 0.22 | 0.19 | 0.15 | 0.07 | −0.09 | −0.57 | $-10^7$ | 1.39 | 0.90 | 0.74 | 0.66 | 0.61 |
| 0.50 | 0.23 | 0.20 | 0.16 | 0.08 | −0.08 | −0.57 | $-10^7$ | 1.41 | 0.92 | 0.76 | 0.68 | 0.63 |
| 0.55 | 0.25 | 0.22 | 0.17 | 0.09 | −0.07 | −0.57 | $-10^7$ | 1.45 | 0.95 | 0.78 | 0.70 | 0.65 |
| 0.60 | 0.27 | 0.24 | 0.20 | 0.11 | −0.05 | −0.56 | $-10^7$ | 1.49 | 0.98 | 0.81 | 0.73 | 0.68 |
| 0.65 | 0.31 | 0.28 | 0.23 | 0.14 | −0.02 | −0.54 | $-10^7$ | 1.54 | 1.02 | 0.85 | 0.77 | 0.72 |
| 0.70 | 0.35 | 0.32 | 0.27 | 0.19 | 0.02 | −0.50 | $-10^7$ | 1.58 | 1.07 | 0.90 | 0.81 | 0.76 |
| 0.75 | 0.40 | 0.37 | 0.33 | 0.25 | 0.08 | −0.42 | $-10^7$ | 1.61 | 1.11 | 0.94 | 0.86 | 0.81 |
| 0.80 | 0.47 | 0.45 | 0.40 | 0.33 | 0.17 | −0.30 | $-10^7$ | 1.61 | 1.13 | 0.98 | 0.90 | 0.85 |
| 0.85 | 0.56 | 0.54 | 0.50 | 0.44 | 0.30 | −0.12 | $-10^7$ | 1.56 | 1.14 | 1.00 | 0.94 | 0.90 |

Table 7

The function $K(\varrho)$ of perfect imaging of spatial frequency $\varrho = R\lambda c$ in multiples of $\lambda c^2$

| $\varrho$ | $D_0(\varrho)$ | $T_1(\varrho)$ | $K(\varrho)$ |
|---|---|---|---|
| 0.01 | 0.987 | 0.00191 | 40.66 |
| 0.02 | 0.974 | 0.007382 | 20.55 |
| 0.03 | 0.962 | 0.01604 | 13.85 |
| 0.04 | 0.949 | 0.02753 | 10.49 |
| 0.05 | 0.936 | 0.04155 | 8.49 |
| 0.10 | 0.875 | 0.13793 | 4.49 |
| 0.15 | 0.810 | 0.25537 | 3.18 |
| 0.20 | 0.747 | 0.36903 | 2.55 |
| 0.25 | 0.685 | 0.4623 | 2.18 |
| 0.30 | 0.624 | 0.52546 | 1.95 |
| 0.35 | 0.561 | 0.5544 | 1.80 |
| 0.40 | 0.505 | 0.54988 | 1.71 |
| 0.45 | 0.425 | 0.5156 | 1.62 |
| 0.50 | 0.391 | 0.45816 | 1.65 |
| 0.55 | 0.337 | 0.3851 | 1.67 |
| 0.60 | 0.285 | 0.30491 | 1.73 |
| 0.65 | 0.235 | 0.2252 | 1.83 |
| 0.70 | 0.188 | 0.1529 | 1.98 |
| 0.75 | 0.144 | 0.09319 | 2.23 |
| 0.80 | 0.104 | 0.04877 | 2.61 |
| 0.85 | 0.0681 | 0.0202 | 3.28 |
| 0.90 | 0.0374 | 0.005503 | 4.67 |

we get

$$\left| A_1 + \frac{a(\varrho)}{2T_1(\varrho)} \right| \leqq B(\varrho) \tag{47}$$

if $B(\varrho) \geqq 0$.

By virtue of (36) this expression can be transformed into

$$|x_0 - \bar{x}_0(\varrho)| \leqq \theta(\varrho), \tag{48}$$

where

$$\theta(\varrho) = 4B(\varrho)|\Delta x_{m,k}|. \tag{49}$$

Hence, analogically to (32) $\theta(\varrho)$ may be written in form

$$\theta(\varrho) = K(\varrho)\lambda c^2 \sqrt{1 - \left[\frac{\Delta x_{m,k}}{{}^{+}\Delta x_{m,k}(\varrho)}\right]^2}, \tag{50}$$

where

$$K(\varrho) = 3.2\,\frac{D_0(\varrho)}{T_1(\varrho)} \tag{51}$$

is considered as the tolerance of defect of focus in the absence of aberrations [3], [10].

It is evident from (50) that the system without spherical aberration ($\Delta x_{m,k} = 0$) implies in accordance with the results in [3] and [10] the maximum interval of defect of focus $2\theta(\varrho) = 2K(\varrho)\lambda c^2$. On the contrary, in systems with $\Delta x_{m,k} = {}^{+}\Delta x_{m,k}(\varrho)\ 2\theta(\varrho) = 0$, i.e., there is only one plane fulfilling the condition of perfect imaging of the spatial frequency $R = \dfrac{\varrho}{\lambda c}$. The values of function $K(\varrho)$, which can be approximated by simple algebraic functions [3], [10], in small intervals of spatial frequency $\varrho$, are given in Table 7.

The permissible interval of defect of focus corresponding to the perfect imaging of a point (30) or a spatial frequency (48) is of practical importance not only for the imaging by monochromatic light, but also for the correction of the spherical aberration in polychromatic light. This problem will be discussed in the following part of this study.

**Tolérance du défaut du foyer en présence de l'aberration sphérique**

On a donné de simples expressions analytiques décrivant la situation dans le voisinage du plan de la meilleure imagerie pour la condition d'imagerie idéale du point illuminé d'une façon incohérente et par rapport aux fréquences spatiales.

**Допуск дефекта фокуса при наличии сферической аберрации**

В работе приведены простые аналитические выражения, описывающие положение в соседстве плоскости наилучшего отображения для условия отображения идеальной точки, некогерентно освещенной, а также в подходе пространственных частот.

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