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# Holographic zone plate as an optical imaging element

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A holographic zone plate has proper spatial frequency characteristics for use as a conventional optical lens. When illuminated by plane or spherical wave front of monochromatic light, produces only one real and one virtual point image. Such a holographic element for imaging purposes is considered, but its aberrations can only be minimized either by proper selection of the recording geometry, or by using special, disturbed recording wave fronts. As a result, we obtained the Newton's equation that is a consequence of the treatment applied to a configuration to achieve the exact Fourier transform relationship between the amplitude distribution in the input and output planes.

## 1. Introduction

There exists a similarity between the optical properties of a transmission zone plate and a converging lens, therefore the possibility of its utilization in optical systems should be fully shown. The characteristic property of a zone plate lies in the fact that the light rays transmitted through the transparent zones reinforce each other, because the path from a point on the optical axis to the first zone is one wavelength shorter than the path to the third zone, and similarly for all adjacent transparent zones [1], [2]. It should be noted that a zone plate is a particular case of an artificially constructed point hologram with the specified characteristic of a positive and a negative primary focal lengths. But the zone plate has not only the primary foci. We can see here a number of higher order foci corresponding to powers which are odd integral multiples of the primary power. One can compare the zone plate to a square topped wave that has a large number of Fourier coefficients, therefore the sharper the boundaries in a zone plate, the greater number of orders are obtained. The holographic zone plate, on the other hand, produced during interference of two wave fronts by a continuous tone of photographic process, does not have sharp alternations, but only the sinusoidal variations. Thus, the intensity of the two interfering waves in emulsion should be arranged such that the information could be recorded on the linear portion of the amplitude transmission against the exposure characteristic. It will be noted that the greater part of the diffracted waves is concentrated in the first order diffraction pattern.

## 2. Holographic zone plate

The holographic zone plate is generated by the interference between a divergent spherical wave front and a reference one, that is usually a plane wave front originated from an axial point at infinity. The spacing of the interference maxima varies with position across the recording medium, and contains enough information to provide a reconstruction of the object point. Such a holographic plate has a sinusoidal amplitude transmittance, and produces one virtual and one real image point. It can be noticed that each zone plate made by the interference of two beams works as a sinusoidal diffraction grating which gives rise only to plus one and minus one diffraction order. Thus, the sinusoidal zone plate differs markedly from the Fresnel zone plate, since it has only one positive and one negative focal length. In the case of interfering a divergent object wave front with a collimated reference wave front in the recording medium, we obtain the phase function of the zone plate in the form

$$\Phi(r) = \frac{2\pi}{\lambda}\left[(r^2+f^2)^{1/2}-f\right] \tag{1}$$

where the focal length $f$ is equal to the distance of the object point from the recording plate; $r = \sqrt{x^2+y^2}$ is the radial coordinate defining the distance of the point on the hologram from the optical axis, whereas the radius of constant phase will be obtained by equating the phase variation (1) to $2\pi n$, *i.e.*,

$$r_n = \sqrt{n^2\lambda^2+2n\lambda f}.$$

If the reference wave originates at a source point located on the optical axis at infinity and the object divergence wave emerges at a point located at a finite distance from the photographic plate, then the amplitude transmittance of the holographic zone plate that is a linear function of the incident light intensity takes the form of a square function of radial coordinate

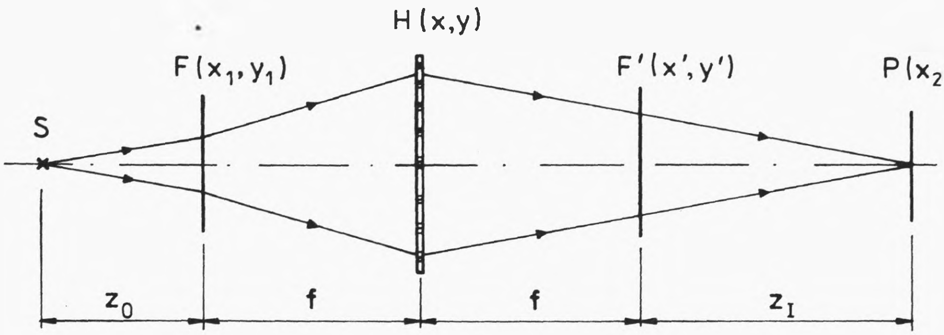
$$t(r) = C\left[A_r^2+\frac{A_o^2}{f^2}+2\frac{A_rA_o}{f}\cos\left(\frac{\pi r^2}{\lambda f}\right)\right]$$

where $A_r$ and $A_o/f$ are the amplitudes of the reference and object waves at a point of the photographic plate, respectively; $C$ is a constant for a given exposure. Due to the nonlinear effects of the photographic emulsion, one can obtain a generalized holographic zone plate which is distorted in such a way that the effects of nonlinearity not only truncate the crests and flatten the valleys of the transmission function, but also squeeze and enlarge the zone areas alternately [3]. In this case, in addition to the principal one, higher order foci will be retrieved that are located at $f/2, f/3, f/4, f/5, \dots$, *etc.*, whereas in the conventional Fresnel zone plate the higher order foci are located at positions of $f/3$, $f/5$, $f/7$, $\dots$, *etc.*

## 3. Imaging equations

It is known that one of the most useful properties of a converging lens is its ability to perform two dimensional Fourier transformation [4]. For ideal Fourier transform lens, the transforming property does not depend on the aperture and field angles but is a space invariance. However, for the real optical systems it varies from lens to lens. The holographic zone plate is equivalent to a conventional optical lens, therefore it can be used for Fourier the transform realization. Analogically, as optical systems are corrected by introducing a plural spherical lens combination, the aberrations of the holographic zone plate are corrected by introducing a set of spherical surfaces whose principal planes are arranged identically [5]. This leads to generalized phase variation in the holographic zone plate; thus we have

$$\Phi_N(r) = \frac{2\pi}{\lambda}\sum_{n=1}^{N}\left[(r^2+f_n^2)^{1/2}-f_n\right]. \tag{2}$$




Schematic diagram of ray tracing through a holographic zone plate

In the paper, we consider the Fourier transform operation, the schematic diagram of which is shown in the Figure. The holographic zone plate working as a converging lens does not perform the Fourier transform in the coherent light. This is achieved by diffraction at the input aperture [4]. Therefore, let a plane object with the amplitude transmittance $t(x_1, y_1)$ be inserted in the front focal plane $F(x_1, y_1)$ of the converging holographic zone plate, and the point source at point $S$ on the optical axis at the distance $z_0$ from the focal plane, as shown in the Figure. The plane $P(x_2, y_2)$ placed at the distance $z_I$ from the back focal plane in the imaging space is the observation plane where the Fourier transform occurs. The point source located on the optical axis emits a diverging spherical wave which gives rise to disturbance at any point in the focal plane $F(x_1, y_1)$. A portion of the spherical wave front is collected by the object transparency and by the holographic optical element that transforms it in the form of a spherical wave converging towards the appropriate point in the observation plane $P(x_2, y_2)$. Using the paraxial approximation, the disturbance at any point in the focal plane $F(x_1, y_1)$ is given by

$$U(x_1, y_1) = \exp\left[i\frac{k}{2z_O}(x_1^2+y_1^2)\right]$$

where the constant phase factor has been omitted; the wave number of propagating wave is $k = \frac{2\pi}{\lambda}$. If the transparence is illuminated by the monochromatic spherical wave of the complex amplitude $U(x_1, y_1)$, then the amplitude distribution just behind the object plane can be written as

$$U_1(x_1, y_1) = t(x_1, y_1)\exp\left[\frac{ik}{2z_O}(x_1^2+y_1^2)\right].$$

Using the Fresnel formula, the amplitude distribution across the holographic zone plate takes the form

$$U(x, y) = A(z)\exp\left[\frac{ik}{2f}(x^2+y^2)\right]\iint\limits_{-\infty}^{\infty} U_1(x_1, y_1)\exp\left[\frac{ik}{2f}(x_1^2+y_1^2)\right]$$
$$\times\exp\left[-i\frac{k}{f}(xx_1+yy_1)\right]dx_1dy_1. \quad (3)$$

Thus, in consequence of phase transformation in the optical element, the amplitude distribution behind the holographic zone plate may be written as

$$U'(x, y) = U(x, y)\exp\left[\frac{ik}{2f}(x^2+y^2)\right].$$

Therefore, we have

$$U'(x, y) = A(z)\iint\limits_{-\infty}^{\infty} U_1(x_1, y_1)\exp\left[\frac{ik}{2f}(x_1^2+ y_1^2)\right]\exp\left[-i\frac{k}{f}(xx_1+yy_1)\right]dx_1dy_1.$$

Applying the Fresnel diffraction integral in the image space, we calculate the amplitude distribution of light at the observation plane

$$U(x_2, x_2) = A'(z)\exp\left[\frac{ik(x_2^2+y_2^2)}{2(f+z_I)}\right]\iint\limits_{-\infty}^{\infty} U'(x, y)\exp\left[\frac{ik(x^2+y^2)}{2(f+z_I)}\right]$$
$$\times\exp\left[-i\frac{k(xx_2+yy_2)}{f+z_I}\right]dxdy,$$

or

$$U(x_2, x_2) = A'(z)\exp\left[\frac{ik(x_2^2+y_2^2)}{2(f+z_I)}\right]\iint\limits_{-\infty}^{\infty} U_1(x_1, y_1)\exp\left[\frac{ik}{2f}(x_1^2+y_1^2)\right]dxdy$$

$$\times\iint_{-\infty}^{\infty}\exp\left[\frac{ik(x^2+y^2)}{2(f+z_I)}\right]\exp\left\{-ik\left[x\left(\frac{x_1}{f}+\frac{x_2}{f+z_I}\right)+y\left(\frac{y_1}{f}+\frac{y_2}{f+z_I}\right)\right]\right\}dxdy \tag{4}$$

where the amplitude and phase of light at coordinates

$$x_2'=\left(1+\frac{z_I}{f}\right)x_1+x_2, \qquad y_2'=\left(1+\frac{z_I}{f}\right)y_1+y_2$$

are related to the amplitude and phase of zone plate spectrum frequencies

$$u=\frac{1}{\lambda}\left(\frac{x_1}{f}+\frac{x_2}{f+z_I}\right), \qquad v=\frac{1}{\lambda}\left(\frac{y_1}{f}+\frac{y_2}{f+z_I}\right).$$

Calculating the Fourier transform of the quadratic phase factor represented by the second integral of Eq. (4), we have

$$\exp\left\{-i\frac{k}{2}(f+z_I)\left[\left(\frac{x_1}{f}+\frac{x_2}{f+z_I}\right)^2+\left(\frac{y_1}{f}+\frac{y_2}{f+z_I}\right)^2\right]\right\}=$$

$$=\exp\left[-i\frac{k}{2}\frac{(f+z_I)}{f^2}(x_1^2+y_1^2)\right]\exp\left[-i\frac{k}{2(f+z_I)}(x_2^2+y_2^2)\right]\exp\left[-i\frac{k}{f}(x_1x_2+y_1y_2)\right]. \tag{5}$$

Thus, substituting expression (5) in Eq. (4), the quadratic phase factors depending on the coordinates $(x_2, y_2)$ are seen to cancel out, leaving Eq. (4) in the form

$$U(x_2,y_2)=A'(z)\iint_{-\infty}^{\infty}t(x_1,y_1)\exp\left\{i\frac{k}{2}\left[\left(\frac{1}{z_O}-\frac{z_I}{f^2}\right)(x_1^2+y_1^2)\right]\right\}$$

$$\times\exp\left[-i\frac{k}{f}(x_1x_2+y_1y_2)\right]dx_1dy_1. \tag{6}$$

The above equation shows that the complex amplitude $U(x_2, y_2)$ is the Fourier transform of the object transmittance modified by the phase factor depending on the position of the observation plane related to the position of the point source. When the phase curvature vanishes, we obtain the exact Fourier transform relationship at the spatial frequencies

$$\xi=\frac{x_2}{\lambda f}, \qquad \eta=\frac{y_2}{\lambda f}$$

between the amplitude distribution in the observation plane and the object transmittance in the front focal plane. But the expression for phase curvature is seen to cancel out, when

$$\frac{1}{z_O}-\frac{z_I}{f^2}=0.$$

The condition leads to the Newton's equation, from which it folllows that for fixed object and image planes, we have

$$z_O z_I = f^2. \tag{7}$$

In our case, the two conjugate planes coincide with the point source and its image in the observation plane. It is evident that the conjugate object and image locations are often given by the respective distances $z_O$ and $z_I$ from the front and the back focal points, as shown in the Figure. As a result, when Newton's equation is satisfied, the Fourier transform operation between the front focal plane and the observation plane can be performed.

## 4. Conclusions

We have shown that the holographic zone plate is equivalent to a conventional imaging lens. The ray tracing originated at a point source on the optical axis, propagating as a divergent wave through the object transparence and the holographic zone plate, has been demonstrated. The considerations carried out in the paraxial region have proved that the exact Fourier transform operation between the front focal plane and the observation plane can be achieved when the Newton's equation for the illuminating source point and its image is valid.

### References

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