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# Gauge reduction of the Fourier transform setup

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The optical Fourier transforming systems are generally realized through the use of an optical focusing element with input and output planes being two focal lengths apart. In this paper, we consider the alternate setups with the reduced longitudinal space.

## 1. Introduction

In a coherently illuminated holographic focusing element there exists a Fourier transform relation between the light amplitude distributions in its front and back focal planes [1]. Due to the transformable property of such a holographic optical element, it is possible to produce the Fourier spectrum with minimum phase error of an arbitrary amplitude transmittance introduced in the input focal plane of the lens. But the Fourier transform relationship can also be realized with minimum phase error in a shorter setup that uses one or two lenses and is one-half the overall length of the generally used conventional spectrum analyzer.

In this paper, we consider three different configurations of Fourier transform relation, whereas each of them is realized in the reduced longitudinal space. As we know, the Fourier transform lens must focus the bundles of parallel light rays diffracted at different angles to suitable points in the back focal plane of the lens. Corresponding to different diffraction angles, the focused spots at all locations in the Fourier plane should be the same. All departures in the quality of the focused spots are the aberrations, and are found as an incorrect reading of spatial frequencies of the Fourier spectrum.

## 2. Focusing mirror transformer

In the case of a holographic mirror, the light used to illuminate the holographic optical element for reconstruction is reflected (diffracted) to form the spherical focusing wave fronts. The design uses the reflecting holographic element in order to achieve a compact folded geometry with high diffraction efficiencies over the desired field of view. Analogously to conventional concave mirror, holographic reflecting element of a given focal length can have different shape factors depending on the recording parameters of the hologram. Thus, if the focal length is defined by

$$\frac{1}{f} = \frac{1}{R_O} - \frac{1}{R_R},$$

then the shape factor is

$$Q = \frac{R_R + R_O}{R_R - R_O}$$

where $R_O$, $R_R$ are the object point and the reference point source distances from the recording medium, respectively. Let the holographic reflecting element be recorded by an object spherical and a reference plane wave fronts, according to the geometry shown in Fig. 1. The input plane wave front is tilted at an $\alpha_R = 6°$ angle with the recording plane and the object beam incident from the right side at $\alpha_O = 0°$ is converged towards the object point at $R_O = 100$ mm from the HOE being formed. It is necessary to expect that the optimum value of the shape is then $Q = 1.0$ corresponding to the plane reference wave ($R_R \to \infty$) and the spherical object wave front whose curvature radius is equal to the focal length ($f = 100$ mm) of the holographic optical element.

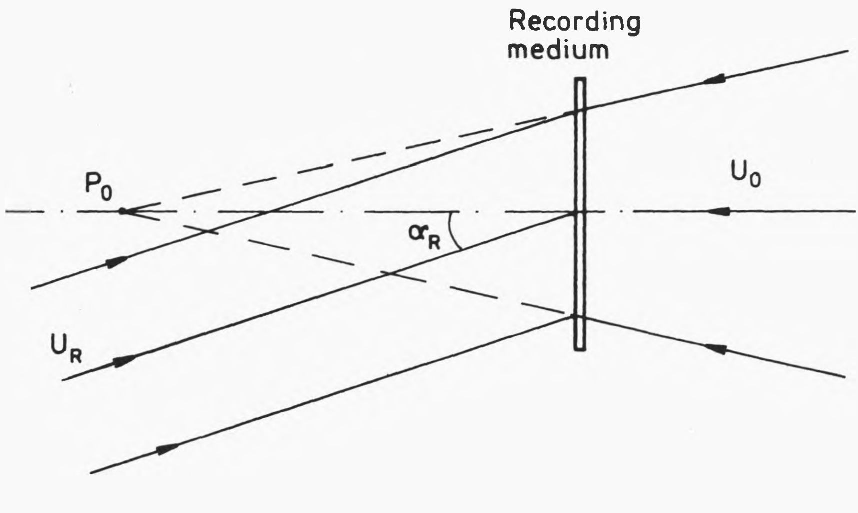


Fig. 1. Recording of an off-axis reflecting holographic element: $\alpha$ is an offset angle, $U_O$, $U_R$ are the complex amplitudes of the object and reference point sources, respectively

It is well known that the third order aberrations of point source hologram were derived by MEIER [2] and CHAMPAGNE [3]. Using the Champagne approach, the plane wave components of the input are being considered. Assuming no wavelength shift, the Gaussian image position is given by the equation

$$\frac{1}{R_I} = \frac{1}{R_C} \pm \left(\frac{1}{R_O} - \frac{1}{R_R}\right) \tag{1}$$

and

$$\sin\alpha_I = \sin\alpha_C \pm (\sin\alpha_O - \sin\alpha_R),$$
$$\cos\alpha_I \sin\beta_I = \cos\alpha_C \sin\beta_C \tag{2}$$

where subscripts $C$ and $I$ refer to the reconstruction source and image points, respectively. But, when the plane wave components of the input are being analyzed, the reconstruction wave has $R_C \to \infty$. Thus the Gaussian image position reduces to $R_I = R_O$, and $\sin\alpha_I = \sin\alpha_C - \sin\alpha_R$. As we know, the third order wave front deviation from the Gaussian sphere in Champagne approximation is determined by

$$\Delta = -\frac{1}{8\lambda}(x^2+y^2)S + \frac{1}{2\lambda}(x^2+y^2)(xC_x + yC_y) - \frac{1}{2\lambda}(x^2A_x + xyA_{xy} + y^2A_y) \tag{3}$$

where the coefficients: $S$, $C_x$, $C_y$, $A_x$, $A_{xy}$, $A_y$ denote the spherical, comatic and astigmatic aberrations, respectively, and are given by:

$$S = \frac{1}{R_C^3} \pm \left(\frac{1}{R_O^3} - \frac{1}{R_R^3}\right) - \frac{1}{R_I^3},$$

$$C_x = \frac{\sin\alpha_C}{R_C^2} \pm \left(\frac{\sin\alpha_O}{R_O^2} - \frac{\sin\alpha_R}{R_R^2}\right) - \frac{\sin\alpha_I}{R_I^2},$$

$$C_y = \frac{\cos\alpha_C \sin\beta_C}{R_C^2} - \frac{\cos\alpha_I \sin\beta_I}{R_I^2},$$

$$A_x = \frac{\sin^2\alpha_C}{R_C} \pm \left(\frac{\sin^2\alpha_O}{R_O} - \frac{\sin^2\alpha_R}{R_R}\right) - \frac{\sin^2\alpha_I}{R_I},$$

$$A_{xy} = \frac{\sin\alpha_C \cos\alpha_C \sin\beta_C}{R_C} - \frac{\sin\alpha_I \cos\alpha_I \sin\beta_I}{R_I},$$

$$A_y = \frac{\cos^2\alpha_C \sin^2\beta_C}{R_C} - \frac{\cos^2\alpha_I \sin^2\beta_I}{R_I}. \tag{4}$$

In our case, the third aberration coefficients become

$$S = \frac{1}{R_O^3} - \frac{1}{R_I^3},$$

$$C_x = -\frac{\sin\alpha_C - \sin\alpha_R}{R_O^2},$$

$$C_y = \frac{\cos\alpha_C \sin\beta_C}{R_O^2},$$

$$A_x = -\frac{(\sin\alpha_C - \sin\alpha_R)^2}{R_O},$$

$$A_{xy} = \frac{(\sin\alpha_C - \sin\alpha_R)\cos\alpha_C \sin\beta_C}{R_O},$$

$$A_y = -\frac{\cos^2\alpha_C \sin^2\beta_C}{R_O}. \tag{5}$$

As shown, for a given specification the third order coma is inversely proportional to the square of the focal length, and third order astigmatism is inversely proportional to the focal length of the holographic element under consideration. Therefore, we see that the performance of the holographic optical element is heavily dependent on its focal length, and can be optimized by increasing the focal length.

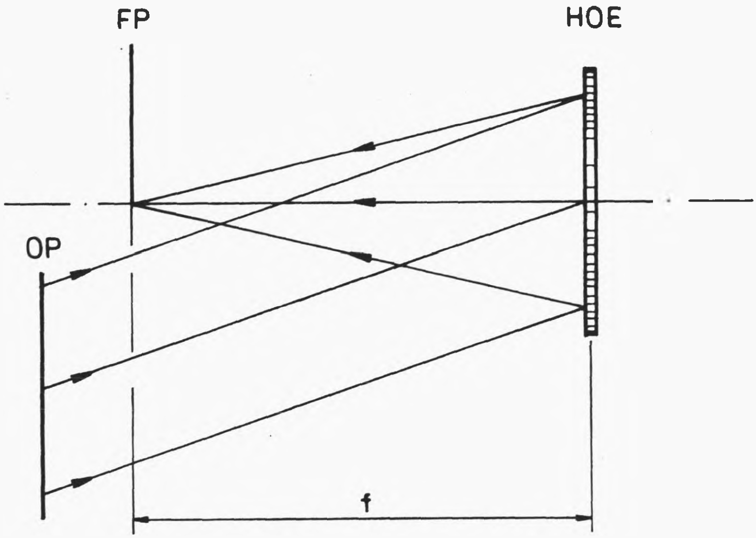


Fig. 2. Holographic reflecting element (HOE) as a Fourier transformer: $f$ – focal length, OP – object plane, FP – Fourier plane

Due to the Fourier transformable property of holographic optical elements, it is possible to form the Fourier spectrum of an arbitrary object transmittance. Usually, the Fourier transform relationship is realized between the input and output planes being two focal lengths apart, but the system with a holographic reflecting element is one-half the overall length, as shown in Fig. 2. The transparency located at a distance $f$ in front of the holographic focusing element is illuminated by a normally incident plane wave. Thus, since the input is inserted in the front focal plane of focusing mirror, the phase curvature disappears leaving an exact Fourier transform of the object transmittance $U_O(x_O, y_O)$ in the Fourier plane: $U_F(x_f, y_f) = F\{U_O(x_O, y_O)\}$.

## 3. Double holographic transformer

Consider an alternate optical setup of two holographic optical elements $H_1$ and $H_2$ that perform a mathematical operation between the input and output planes being a focal length apart. This system is shown in Fig. 3, where a plane object with amplitude transmittance $U_O(x_O, y_O)$ is inserted immediately in front of the first

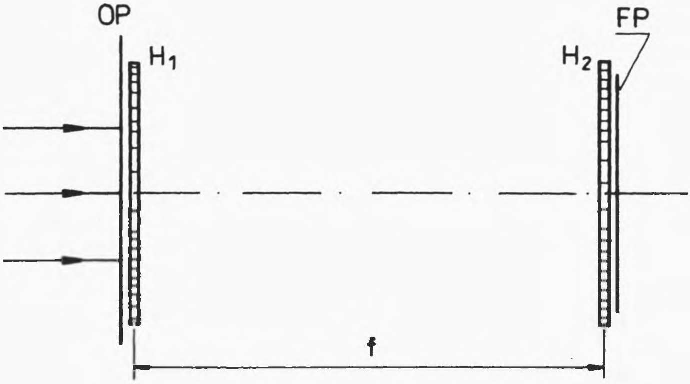


Fig. 3. Double holographic lens transformer ($H_1$, $H_2$). Object plane OP is placed in the ($x_O, y_O$) and the Fourier plane FP — in the ($x_f, y_f$) planes of the coordinate system

holographic converging element $H_1$ of focal length $f$. The object is illuminated by a normally incident, coherent plane wave. To find the amplitude distribution across the focal plane of the first holographic lens, we use the Fresnel diffraction formula [1] that in our case can be written in the form

$$U_F(x_f,y_f) = \frac{\exp\left[i\frac{k}{2f}(x_f^2+y_f^2)\right]}{i\lambda f}\iint\limits_{-\infty}^{+\infty} U_O(x_O,y_O)\exp\left[-i\frac{k}{f}(x_0x_f+y_Oy_f)\right]dx_Ody_O. \quad (6)$$

We see that the amplitude distribution in the output plane is proportional to the Fourier transform of the object amplitude transmittance. But as we know, the phase transformation by an optical converging element of the focal length $f$ at co-ordinates $(x,y)$ may be written in the form of quadratic approximation to a spherical wave

$$\exp\left[-i\frac{k}{2f}(x^2+y^2)\right].$$

Therefore, when the second holographic lens of focal length $f$ is inserted in the back focal plane of the first lens, the quadratic phase factor is seen to cancel, leaving an exact Fourier transform relation

$$U_F(x_f,y_f) = \frac{1}{i\lambda f}\iint\limits_{-\infty}^{+\infty} U_O(x_O,y_O)\exp\left[-i\frac{k}{f}(x_Ox_f+y_Oy_f)\right]dx_Odx_O. \quad (7)$$

We see that the double holographic lens system can be used as a Fourier transform setup, where the first holographic element performs the Fourier transform with phase curvature and the second one acts as a phase corrector plate. Thus, the double lens setup will require only half the longitudinal space of the conventional one.

## 4. Converging beam Fourier transform

The converging beam Fourier transform is based on the diffraction of a converging spherical wave by the object frequency components [4], as shown in Fig. 4. In this case, object is placed behind the holographic lens which is corrected for spherical aberration to produce a perfect spherical wave. If the holographic optical element is illuminated by a normally incident plane wave of an amplitude $A$, then it is the

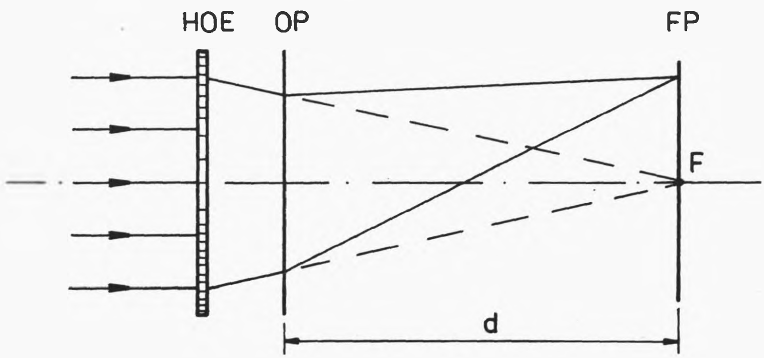


Fig. 4. Converging beam setup for Fourier transform

spherical wave that is incident on the object, converging towards the focus of the lens, and the field distribution across the focal plane is proportional to Fourier transform of the object transmittance

$$U_F(x_f,y_f) = \frac{A\exp\left[i\frac{k}{2d}(x_f^2+y_f^2)\right]}{i\lambda d}\iint_{-\infty}^{+\infty} U_O(x_O,y_O)\exp\left[-i\frac{k}{d}(x_O x_f+y_O y_f)\right]dx_O dy_O. \quad (8)$$

Thus, up to a quadratic phase factor, the field distribution in the back focal plane is the Fourier transform of that portion of the object subtended by the projected lens aperture. The setup allows the size of spectrum to be varied, because of the spatial frequencies that are given by

$$\xi = \frac{x_f}{\lambda d}, \quad \eta = \frac{y_f}{\lambda d}.$$

We see that by increasing the distance between the object and Fourier planes, the spatial size of the transform is increased, and by decreasing the distance it is made smaller. Independently of the corrected spherical wave for an on-axis point, aberrations in the Fourier plane are induced by this wave owing to diffraction at the frequency components of the object. Therefore, it is important to know these aberrations quantitatively. Using the Meier's formulas [2] of the third-order holographic aberrations for linear holographic grating that is inserted in the object plane, we have for spherical aberration $\Delta_S = 0$, the wave front deviation

$$\Delta = \Delta_C + \Delta_A + \Delta_F$$

where

$$\Delta_C = -\frac{\lambda\xi}{2d^2} R^3 \cos\theta,$$

$$\Delta_A = \frac{\lambda^2\xi^2}{2d} R^2 \cos^2\theta,$$

$$\Delta_F = \frac{\lambda^2\xi^2}{4d} R^2 \tag{9}$$

are the wave front deviations owing to coma, astigmatism and field curvature, respectively. $\xi$ is the spatial frequency of the holographic grating, and $R$, $\theta$ are the polar coordinates in the exit pupil of the system, whereas the angle $\theta$ is formed by $R$ with the positive $y$-direction. The fifth aberration, that is, distortion yields only a shift of image in the Fourier plane.

In this case, the Fourier transform relation between the object transmittance and the amplitude distribution in the focal plane is not an exact one, due to the presence of the quadratic phase factor (see Eq. (8)). But in most cases the intensity distribution across the focal plane is measured that yields knowledge of the power spectrum of the object transmittance

$$|U_f(x_f, y_f)|^2 = \frac{A^2}{\lambda^2 d^2}\left|\iint_{-\infty}^{+\infty} U_O(x_O, y_O)\exp\left[-i\frac{k}{d}(x_O x_f + y_O y_f)\right]dx_O dy_O\right|^2.$$

## 5. Conclusions

The three different configurations with the reduced longitudinal space for Fourier transform setup are considered. Note that two of them realize an exact Fourier transform relation between the object transparency and the amplitude distribution across the focal plane, and the third one is represented by Fourier transform relation with the presence of the quadratic phase factor that precedes the integral presenting the modified Fresnel diffraction formula.

### References

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