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Holograms of corrected spherical and comatic aberrations**

In the paper the conditions for simultaneous correction of spherical and comatic aberrations have been given for chosen position of reference wave source with respect to object position for definite μ/m ratio.

1. Introduction

The problem of hologram aberrations of third order has been so far analysed by several authors. The basic work, in which the fundamental relations enabling evaluation of the spherical aberration, coma, field curvature, astigmatism and distortion are reported, is that by Meier [1]. In this paper the simplest cases of correction of the said abberations are also discussed. The conditions of aberration correction may be found also in the paper Champagne [2].

In the present paper we want to give an exact analysis of simultaneous correction of spherical and comatic aberrations for both the holographic images.

2. Theory

Let us introduce the following notation in accordance with fig. 1:

$P(x_1, y_1, z_1)$ — the object position.

$R(x_r, y_r, z_r)$ — the position of the reference wave source,

$C(x_c, y_c, z_c)$ — the position of the reconstructing wave source,

$(X_2', 0', Y_2')$ — the hologram plane during recording.

$(X_2, 0, Y_2)$ — the hologram plane during reconstruction,

z_3 — the images distance from the hologram.

The object distance is determined by formula [1]:

$$\frac{1}{z_3} = \frac{1}{z_c} \pm \frac{\mu}{m^2 z_1} \mp \frac{\mu}{m^2 z_r} \tag{1}$$

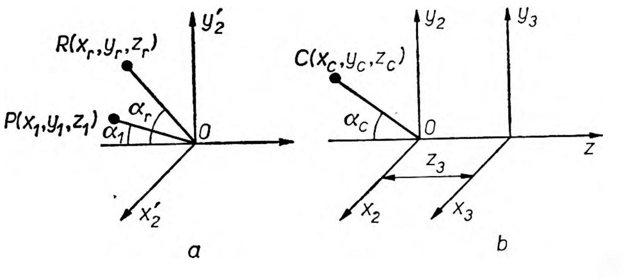

Fig. 1. a) Hologram recording setup; b) hologram reconstruction setup

where

$$m = \frac{x_2}{x_2'} = \frac{y_2}{y_2'}, \tag{2}$$

$$\mu = \frac{\lambda_2}{\lambda_1} \tag{3}$$

(λ_1 — the wavelength of light used for recording, λ_2 — the wavelength of light used for reconstruction). The upper sign is attributed to the primary image, while the lower sign corresponds to the secondary one. This convention is valid thoughout the whole paper; eventual deviations being always marked.

The following expressions are the counterparts of the first and second Seidel sums in the classical optics [2]:

$$S_1 = \frac{1}{z_c^3} \pm \frac{\mu}{m^4}\left(\frac{1}{z_1^3} - \frac{1}{z_1^3}\right) - \frac{1}{z_3^3}, \tag{4}$$

$$S_2 = \frac{x_c}{z_c} \pm \frac{\mu}{m^3}\left(\frac{x_1}{z_1^3} - \frac{x_r}{z_r^3}\right) - \frac{x_3}{z_3^3}, \tag{5}$$

where

$$x = \frac{m^2 x_c z_1 z_r \pm \mu m x_1 z_c z_r \mp \mu m x_r z_c z_1}{m^2 z_1 z_r \pm \mu z_c z_r \mp \mu z_c z_1}. \tag{6}$$

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In the face of (1) and (6) the expressions (4) and (5) may be represented in the form:

$$S_1 = \frac{\mu}{m^4}\left[\mp\left(\frac{1}{z_1^3}-\frac{1}{z_r^3}\right)\left(\frac{\mu^2}{m^2}-1\right)-\right.$$

$$-3\frac{\mu}{z_c}\left(\frac{1}{z_1^2}+\frac{1}{z_r^2}\right)\mp 3\left(\frac{m^2}{z_c^2}-\frac{\mu^2}{m^2}\frac{1}{z_r z_1}\right)\times$$

$$\left.\times\left(\frac{1}{z_1}-\frac{1}{z_r}\right)+\frac{6\mu}{z_c z_1 z_r}\right], \qquad (7)$$

$$S_2 = \frac{x_c}{z_c}\left[\frac{1}{z_c^2}-\left(\frac{1}{z_c}\pm\frac{\mu}{m^2}\frac{1}{z_1}\mp\frac{\mu}{m^2}\frac{1}{z_r}\right)^2\right]\pm$$

$$\pm\frac{x_1}{z_1}\frac{\mu}{m}\left[\frac{1}{m^2 z_1^2}-\left(\frac{1}{z_c}\pm\frac{\mu}{m^2}\frac{1}{z_1}\mp\frac{\mu}{m^2}\frac{1}{z_r}\right)^2\right]\pm$$

$$\pm\frac{x_r}{z_r}\frac{\mu}{m}\left[\frac{1}{m^2 z_r^2}-\left(\frac{1}{z_c}\pm\frac{\mu}{m^2}\frac{1}{z_1}\mp\frac{\mu}{m^2}\frac{1}{z_r}\right)^2\right]. \qquad (8)$$

It is easy verify that the spherical aberration disappears for $z_r = z_c = \infty$ if $\mu = m$, and also for $z_r = z_1$ at arbitrary μ/m ratio and arbitrary z_c. In the first case ($z_r = z_c = \infty$) coma disappears if

$$\frac{x_c}{z_c} = \pm\frac{\mu}{m}\frac{x_r}{z_r}, \qquad (9)$$

and for $z_r = z_1$, z_c it must satisfy the condition

$$z_c = \pm m z_1. \qquad (10)$$

The condition (10) is analogical for both primary and secondary images (the sign $\pm$ corresponds to both primary and secondary images).

Let us analyse the problem in more detailed way. The formulae (7) and (8) may be transformed by using the following notations

$$\frac{x_1}{y_1} = \tan\alpha_1, \quad \frac{x_r}{z_r} = \tan\alpha_r, \quad \frac{x_c}{z_c} = \tan\alpha_c,$$

$$\tan\alpha_1 = q\tan\alpha_r, \ \tan\alpha_c = t\tan\alpha_r, \qquad (11)$$

$$z_c = \frac{z_1}{p}, \quad z_r = \frac{z_1}{r}.$$

Hence, we obtain

$$S_1 = \frac{\mu}{m^4 z_1^3}\left\{\mp 3m^2(1-r)p^2-3\mu(1-r)^2p\mp\right.$$

$$\left.\mp\frac{\mu^2}{m^2}(1-r)^3\pm 1\mp r^3\right\}, \qquad (12)$$

$$S_2 = \frac{\mu}{m^3}\frac{\tan\alpha_r}{z_1^2}\left\{\mp(q-1)m^2p^2+2m(1-r)\times\right.$$

$$\times\left[\mp t-\frac{\mu}{m}(q-1)\right]p-\frac{\mu}{m}(1-r)^2\times$$

$$\left.\times\left[t\pm\frac{\mu}{m}(q-1)\right]\pm q\mp r^2\right\}. \qquad (13)$$

Let us consider first the correction of spherical. To correct the spherical aberration the condition $S_1 = 0$ must be fulfilled, i.e.

$$\mp 3m^2\,(1-r)p^2-3\mu(1-r)^2p\mp\frac{\mu^2}{m^2}(1-r)^3\pm$$

$$\pm 1\mp r^3 = 0. \qquad (14)$$

The roots of equation (14) have the form:

$$p_1 = \frac{\mu}{2m^2}(1-r)+\frac{1}{6m^2}\times$$

$$\times\sqrt{9\mu^2(1-r)^2\pm\frac{12m^2}{1-r}\left[\mp\frac{\mu^2}{m^2}(1-r)^3\mp r^3\pm 1\right]}, \qquad (15a)$$

$$p_2 = \mp\frac{\mu}{2m^2}(1-r)-\frac{1}{6\,m^2}\times$$

$$\times\sqrt{9\mu^2\,(1-r)^2\pm\frac{12\,m^2}{1-r}\left[\mp\frac{\mu^2}{m^2}(1-r)^3\mp r^3\pm 1\right]}. \qquad (15b)$$

To make the solution real the following conditions must be satisfied

$$\frac{\mu^2}{m^2} \leqslant 4\,\frac{1-r^3}{(1-r)^3}, \qquad (16a)$$

for $r < 1$,

$$\frac{\mu^2}{m^2} \geqslant \frac{1-r^3}{(1-r)^3}, \qquad (16b)$$

for $r > 1$.

For $r = 1$ the equation (14) is identically fulfilled. The admitted values of the μ/m ratio for several examplified values of r are given in table 1 (for both images).

Table 1

r	μ/m
-2	$\leqslant 1.15$
-1	$\leqslant 1$
-0.5	$\leqslant 1.15$
0	$\leqslant 2$
0.5	$\leqslant 5.29$
1	arbitrary
2	$\leqslant 5.29$

Now, we solve the equation (14) for several values of r and μ/m. Thus, for illustration we choose the following cases:

I $r = 1$,
II 1 $r = 0$, $\mu/m = 1$,
II 2 $r = 0$, $\mu/m = 2$,
III 1 $r = 0.5$, $\mu/m = 1$,
III 2 $r = 0.5$, $\mu/m = 2$,
IV 1 $r = -1$, $\mu/m = 1$,
IV 2 $r = -1$, $\mu/m = 0.5$.

In the case IV, in accordance with the table 1, $\mu/m \leqslant 1$, therefore another value of the ratio has been chosen than that for cases II and III. For the case I the ratio μ/m, in accordance with the discussion following eq. (14), may be arbitrary. The results are gathered in table 2. In the fourth column the solutions of equation (14) are given, i.e. the condition to be satisfied by the position of the reconstruction wave source z_c to make the spherical aberration disappear. In the sixth column of the table 2 the image positions are given. It may be seen that in case II 1b and IV 2b the image appears at infinity and thus those cases are practically useless.

Now, let us consider the correction of coma. We want coma to be corrected for the cases when spherical aberration diasappears. In accordance with (13) the condition for correction of coma has the form:

$$\mp m^2(q-1)p^2+2m(1-r)\left[\mp t-\frac{\mu}{m}(q-1)\right]p-$$
$$-\frac{\mu}{m}(1-r)^2\left[t\pm\frac{\mu}{n}(q-1)\right]\pm q\mp r^2 = 0. \tag{17}$$

The roots of equation (17) are the following:

$$p_1 = \frac{\pm(1-r)}{m(q-1)}\left[\mp t-\frac{\mu}{m}(q-1)\right]+$$
$$+\frac{1}{m(q-1)}\sqrt{(1-r)^2\left[\mp t-\frac{\mu}{m}(q-1)\right]^2\mp(q-1)\times\left\{\frac{\mu}{m}(1-r)^2\left[t\pm\frac{\mu}{m}(q-1)\right]\mp q\pm r^2\right\}}, \tag{18a}$$

$$p_2 = \pm\frac{(1-r)}{m(q-1)}\left[\mp t-\frac{\mu}{m}(q-1)\right]-$$
$$-\frac{1}{m(q-1)}\sqrt{(1-r)^2\left[\mp t-\frac{\mu}{m}(q-1)\right]^2\mp(q-1)\times\left\{\frac{\mu}{m}(1-r)^2\left[t\pm\frac{\mu}{m}(q-1)\right]\mp q\pm r^2\right\}}. \tag{18b}$$

The results obtained are gathered in the column 5 of table 2. If may be seen that only for cases I, II 1a, III1 b, IV 1, and IV 2a coma is corrected for the whole field and for the both images (in the third order region). In the other cases coma either can not be

Table 2

Notation of analysed cases	μ/m	z_r	Condition of correction of spherical aberration	Condition of coma correction	z_3
I	Arbitrary	z_1	Corrected for both the images	$z_c = \pm m z_1$*	$\pm m z_1$*
II 1a	1	∞	$z_c = \infty$	$\tan\alpha_c = \pm\tan\alpha_r$	$\pm m z_1$
II 1b	1	∞	$z_c = \mp m z_1$	$\tan\alpha_c = \mp\tan\alpha_1$	∞
II 2	2	∞	$z_c = \mp m z_1$	impossible	$\pm m z_1$
III 1a	1	$2 z_1$	$z_c = \mp m z_1$	$\tan\alpha_c = \mp\tan\alpha_1$	$\mp 2 m z_1$
III 1b	1	$2 z_1$	$z_c = \pm 2 m z_1$	$\tan\alpha_c = \pm\tan\alpha_r$	$\pm m z_1$
III 2a	2	$2 z_1$	$z_c = \mp\frac{5}{6} m z_1$	$\tan\alpha_c = \pm 0.3\alpha_r$ $\mp 1.37\times\tan\alpha_1$	$\mp 5 m z_1$
III 2b	2	$2 z_1$	$z_c = \pm 5 m z_1$	$\tan\alpha_c = \pm 1.37\alpha_r$ $\mp 0.63\times\tan\alpha_1$	$\pm\frac{5}{6} m z_1$
IV 1	1	$-z_1$	$z_c = \mp m z_1$	Corrected for both the images	$\pm m z_1$
IV 2a	0.5	$-z_1$	$z_c = \infty$	$\tan\alpha_c = 0$	$\pm m z_1$
IV 2b	0.5	$-z_1$	$z_c = \mp m z_1$	$\tan\alpha_c = \pm 0.5$ $\mp 0.5\times\tan\alpha_1$	∞

*) Sign $\pm$ corresponds in this case to both the primary and secondary images.

corrected, or is corrected only for definite field angle. This correction is of no essential practical importance. We are interested first of all in correcting coma in the whole field for corrected spherical aberration. In order to make some correction independent of field angle the coefficient appearing at q in expression (17) must be equal to zero. Thus, we have the following conditions:

$$\mp m^2p^2-2\mu(1-r)p\mp \frac{\mu^2}{m^2}(1-r)^2\pm 1 = 0. \tag{19}$$

The roots of equation (19) have the form

$$p_1 = \mp \frac{\mu}{m^2}(1-r)+\frac{1}{m}, \tag{20a}$$

$$p_2 = \mp \frac{\mu}{m^2}(1-r)-\frac{1}{m}. \tag{20b}$$

Combination of equations (20) with the condition of spherical aberration correction (15) yields the sought condition in form of the following equations identical for both the images

$$r^2\left(1-\frac{\mu^2}{m^2}\right)+r\left(1+2\frac{\mu^2}{m^2}\pm 3\frac{\mu}{m}\right)-$$
$$-2-\frac{\mu^2}{m^2}\mp 3\frac{\mu}{m} = 0. \tag{21}$$

We solve this equation for r equal to 0.1, 0.5' —1, i.e. for the positions of the reference wave source, which have been considered so far (table 2). The solutions of equation (21) for those cases are given in table 3.

Table 3

No	z_r	μ/m
I	z_1	arbitrary
II	∞	1, 2
III	$2z_1$	1, 5
IV	$-z_1$	1, 0.5

By comparing the results given in tables 2 and 3 we state that, for instance, for $z_r = 2z_1$ and $\mu/m = 2$ (which was considered above) no correction of coma may be achieved for the whole field of view (for corrected spherical aberration). Such correction, at the assumed position of the reference wave sources, may be obtained if $\mu/m = 1$ (III 1b) and $\mu/m = 5$ (the latter case was not considered here).

The purpose of this work was to give a correction analysis for aplanatic type holograms, and consequently of astigmatism correction has not been taken into account. If the field of view is relatively small the respective holograms have practical significance analogical to that of their classical counterparts. However, as the correction of astigmatism is essential, it will be checked in the above discussed cases for simultaneous correction of spherical and comatic aberrations (I, II 1a, III 1b, IV 1, IV 2a).

The correction condition for astigmatism has the form [2]:

$$S_3 = \frac{x_c^2}{z_c^3}\pm\frac{\mu}{m^2}\left(\frac{x_1^2}{z_1^2}-\frac{x_r^2}{z_r^2}\right)-\frac{x_3}{z_3^3}. \tag{22}$$

Inserting the relations (1), (6) and (11) into (22) we get

$$S_3 = \frac{\mu}{m^2}\,\frac{\tan^2\alpha_r}{z_1}\Big\{[-\mu(q-1)^2\mp 2mt(q-1)]p\pm q^2\mp$$
$$\pm r+(1-r)\left[\mp\frac{\mu^2}{m^2}(q-1)^2-2\frac{\mu}{m}t(q-1)\mp t^2\right]\Big\}. \tag{23}$$

It may be easily verified that in the first four cases of aplanatic correction the astigmatism is also corrected, while in the fifth case (IV 2a) astigmatism is different from zero.

3. Conclusions

Summing up it should be noted that there exist a number of possible ways allowing to obtain a simultaneous correction of spherical and comatic aberrations depending on the position of the reference wave source and the value of the μ/m ratio. It appears that in some cases of aplanatic correction the astigmatism is automatically corrected. The next paper will devoted to a more exact analysis of all three aberrations.

Голограммы с корректированной сферической аберрацией и аберрацией комы

Преобразованы формулы, описывающие сферическую аберрацию и кому голограмм, к виду, удобному для нахождения условий параллельной коррекции обеих аберраций для нескольких положений источника волны отсчета.

References

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