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Pupil effect in nonrotation-symmetric gradient-index material*

C. Gómez-Reino

Departamento de Optica, Facultade de Fisica, Universidade de Santiago de Compostela, Spain.

E. Larrea

Asociacion, Central Nuclear de Regodola, La Coruña, Spain.

The effect of the transmittance function in nonrotation-symmetric gradient-index material due to a circular pupil is studied and this material is characterized by its effective transmittance function.

1. Introduction

Recently, fibres and lenses with rotation-symmetric gradient-index profiles have begun to be used in imaging systems, and these imaging capabilities of graded-index materials hold considerable promise for a wide variety of application [1–6]. In earlier papers [7, 8], the authors have studied imaging and transforming in nonsymmetric gradient-index material and obtained image and transform conditions. In the recent paper [9] the authors have also studied pupil effect in symmetric gradient-index material. In this paper, we study the effect on the transmittance function in nonsymmetric gradient-index material, when a circular aperture in situated in the input plane. We assume that the refractive index is given by [8]:

$$n^2(x, y, z) = n_1^2(z) = n_0^2 h_1(z)x + h_2(z)y - g^2(z)(x^2+y^2) \tag{1}$$

where n_0 is the index at the central axis, and n_1, g, h_1 and h_2 are arbitrary functions of z.

2. Pupil effect

Let us consider an inhomogeneous medium with the refractive index given by Eq. (1), limited to planes $z = 0$ and $z = d$, surrounded by a vacuum and with a circular aperture of radius r_0 on the input plane $z = 0$ (Fig. 1). When this

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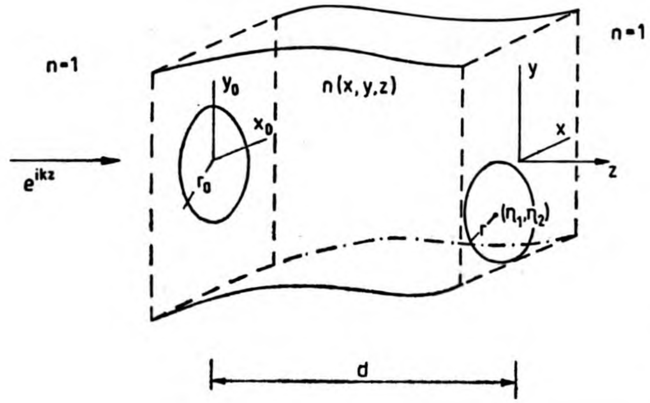

Fig. 1. Pupil effect in nonrotation-symetric gradient-index material

medium is illuminated from the left by a monochromatic plane wave of unit amplitude and wavelength λ, the complex amplitude distribution at a plane $z > 0$ within the nonrotation-symmetric GRIN material can be expressed as [8]

$$a(\varrho, z) = \sqrt{\frac{n_0}{n_1(z)}} \exp\left(ik\int_0^z n_1(z')dz'\right)\exp\left(ikn_0\left\{\dot{\vec{\eta}}(\tau)[\vec{\varrho}-\vec{\eta}(\tau)] + \frac{1}{2}\int_0^\tau L(\tau')d\tau'\right\}\right) \times \int_0^{2\pi}\int_0^{r_0} G(\varrho-\eta(\tau), \varrho_0, \tau)\,\varrho_0 d\varrho_0 d\theta, \tag{2}$$

where

$$\tau(z) = n_0 \int_0^z \frac{dz'}{n_1(z')}, \tag{3}$$

and G is Green's function defined as

$$G = -\frac{ikn_0}{2\pi H_1(\tau)} \exp\left(i\frac{kn_0}{2H_1(\tau)}\{\dot{H}_1(\tau)[\varrho-\eta(\tau)]^2 + H_2(\tau)\varrho_0^2 - 2[\varrho-\eta(\tau)]\varrho_0\cos\theta\}\right), \tag{4}$$

with

$$\vec{\varrho}-\vec{\eta}(\tau) \equiv (x-\eta_1(\tau), y-\eta_2(\tau)), \quad \varrho_0 = \sqrt{x_0^2+y_0^2}, \tag{5}$$

$$\dot{\vec{\eta}}(\tau) \equiv (\dot{\eta}_1(\tau), \dot{\eta}_2(\tau)), \quad L(\tau) = \dot{\eta}_1^2+\dot{\eta}_2^2+h_1^0\eta_1+h_2^0\eta_2-g_0^2(\eta_1^2+\eta_2^2), \tag{6}$$

$$h_{1,2}^0[\tau(z)] \equiv h_{1,2}(z), \quad g_0[\tau(z)] \equiv g(z), \tag{7}$$

and $H_{1,2}(\tau)$ are two independent solutions of the equation

$$\ddot{H}_{1,2}(\tau) + g_0^2(\tau)H_{1,2}(\tau) = 0, \tag{8}$$

with the initial conditions

$$H_1(0) = \dot{H}_2(0) = 0, \quad \dot{H}_1(0) = H_2(0) = 1 \tag{9}$$

where the point denotes the derivative with respect to τ, and the Wronskian is equal to 1, that is

$$\dot{H}_1(\tau)H_2(\tau) - H_1(\tau)\dot{H}_2(\tau) = 1. \tag{10}$$

In a similar way, $\eta_{1,2}(\tau)$ are two solutions of the following equation:

$$\ddot{\eta}_{1,2}(\tau) + g_0^2(\tau)\eta_{1,2}(\tau) = \frac{h_{1,2}^0(\tau)}{2}, \tag{11}$$

with the initial conditions:

$$\eta_{1,2}(0) = \dot{\eta}_{1,2}(0) = 0. \tag{12}$$

From Eqs. (8), (9), (11) and (12), the functions η and H are related by

$$\eta_{1,2}(\tau) = \frac{1}{2}\int_0^\tau [H_1(\tau)H_2(\tau') - H_1(\tau')H_2(\tau)]\, h_{1,2}^0(\tau')d\tau'. \tag{13}$$

On the other hand, this medium is characterized by its transmittance function $t(\varrho, d)$ at the output plane $z = d$ and this function has been defined without aperture by [8]:

$$\begin{aligned} t(\varrho, d) = {} & \sqrt{\frac{n_0}{n_1(d)}} \cdot \frac{1}{H_2[\tau(d)]} \exp\left(ik\left[\int_0^d n_1(z')dz' + \frac{n_0}{2}\int_0^{\tau(d)} L(\tau')d\tau'\right]\right) \\ & \times \exp\left(-i\frac{\pi n_0 H_2[\tau(d)]}{\lambda \dot{H}_2[\tau(d)]}\, \dot{\eta}^2[\tau(d)]\right) \\ & \times \exp\left(i\frac{\pi n_0 \dot{H}_2[\tau(d)]}{\lambda H_2[\tau(d)]}\left[\varrho - \eta[\tau(d)] + \dot{\eta}[\tau(d)]\frac{H_2[\tau(d)]}{\dot{H}_2[\tau(d)]}\right]^2\right). \end{aligned} \tag{14}$$

Equation (14) indicates that the transmittance function may be regarded as first-order approximation to a spherical wave. The wave is converging toward (or diverging from) a point which does not lie on the z axis. The location of this point is given by [10]:

$$x_i = \eta_1[\tau(d)] - \dot{\eta}_1[\tau(d)]\frac{H_2[\tau(d)]}{\dot{H}_2[\tau(d)]}, \tag{15a}$$

$$y_i = \eta_2[\tau(d)] - \dot{\eta}_2[\tau(d)]\frac{H_2[\tau(d)]}{\dot{H}_2[\tau(d)]}, \tag{15b}$$

$$z_i = \frac{H_2[\tau(d)]}{n_0\dot{H}_2[\tau(d)]} \tag{15c}$$

where x_i and y_i are the off-axis coordinates of the focus, and the z coordinate is the distance from the focus to the output plane $z = d$; when $z_i > 0$, the focus is situated to the left of the plane $z = d$; when $z_i < 0$, the focus is situated to the right of the plane $z = d$, as shown in Fig. 2. From Eqs. (15) it follows that the coordinates of the focus depend on the output plane location.

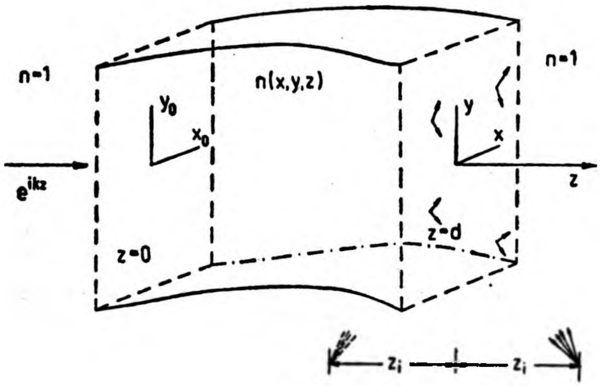

Fig. 2. Transmittance function

If we introduce dimensionless variables

$$t = \frac{\varrho_0}{r_0}, \tag{16a}$$

$$u[\tau(z)] = \frac{kn_0 H_2[\tau(z)]}{H_1[\tau(z)]} r_0^2, \tag{16b}$$

$$v[\tau(z)] = \frac{kn_0\big(\varrho - \eta[\tau(z)]\big)}{H_1[\tau(z)]} r_0, \tag{16c}$$

$$\frac{u[\tau(z)]}{v[\tau(z)]} = \frac{r_0}{\varrho - \eta[\tau(z)]} H_2[\tau(z)], \tag{16d}$$

Equation (2) may be written as

$$a(\varrho, z) = -i\sqrt{\frac{n_0}{n_1(z)}}\,\frac{u[\tau(z)]}{H_2[\tau(z)]}\exp\left(ik\int_0^z n_1(z')\,dz'\right)$$
$$\times\exp\left(ikn_0\left\{\vec{\eta}[\tau(z)][\vec{\varrho}-\vec{\eta}][\tau(z)]+\frac{1}{2}\int_0^\tau L[\tau'(z)]\,d\tau'\right\}\right) \tag{17}$$
$$\times\exp\left(i\,\frac{u[\tau(z)]\dot{H}_1[\tau(z)]}{2H_2[\tau(z)]}\left[\frac{\varrho-\eta[\tau(z)]}{r_0}\right]^2\right)\int_0^1 J_0\big(v[\tau(z)]t\big)\exp\left(i\,\frac{u[\tau(z)]t^2}{2}\right)t\,dt$$

where the integration has been performed on θ and J_0 is the zero-order Bessel function of the first kind.

The earlier diffraction integral may be evaluated in terms of the Lommel functions [11, 12]:

$$a(\varrho, z) = t(\varrho, z) + \frac{i}{H_2[\tau(z)]}\sqrt{\frac{n_0}{n_1(z)}}\,[iV_0(u, v) + V_1(u, v)]$$

$$\times \exp\left(ik\int_0^z n_1(z')dz'\right)\exp\left(ikn_0\left\{\dot{\vec{\eta}}[\tau(z)]\left[\vec{\varrho}-\vec{\eta}[\tau(z)]\right]+\frac{1}{2}\int_0^\tau L[\tau'(z)]d\tau'\right\}\right) \tag{18a}$$

$$\times \exp\left(i\frac{u[\tau(z)]}{2}\left\{\frac{\dot{H}_1[\tau(z)]}{H_2[\tau(z)]}\left[\frac{\varrho-\eta[\tau(z)]}{r_0}\right]^2+1\right\}\right), \quad \text{for } \varrho-\eta[\tau(z)] < r_0H_2[\tau(z)],$$

$$a(\varrho, z) = -\frac{1}{H_2[\tau(z)]}\sqrt{\frac{n_0}{n_1(z)}}\,[U_2(u, v) + iU_1(u, v)]\exp\left(ik\int_0^z n_1(z')dz'\right)$$

$$\times \exp\left(ikn_0\left\{\dot{\vec{\eta}}[\tau(z)]\left[\vec{\varrho}-\vec{\eta}[\tau(z)]\right]+\frac{1}{2}\int_0^\tau L[\tau'(z)]d\tau'\right\}\right) \tag{18b}$$

$$\times \exp\left(i\frac{u[\tau(z)]}{2}\left\{\frac{\dot{H}_1[\tau(z)]}{H_2[\tau(z)]}\left[\frac{\varrho-\eta[\tau(z)]}{r_0}\right]^2+1\right\}\right), \quad \text{for } \varrho-\eta[\tau(z)] > r_0H_2[\tau(z)],$$

$U_{1,2}$ and $V_{0,1}$ being the Lommel functions:

$$U_n(u, v) = \sum_{s=0}^{\infty}(-1)^s\left(\frac{u}{v}\right)^{n+2s} J_{n+2s}(v), \tag{19a}$$

$$V_n(u, v) = \sum_{s=0}^{\infty}(-1)^s\left(\frac{v}{u}\right)^{n+2s} J_{n+2s}(v). \tag{19b}$$

Setting in Eqs. (18) $z = d$, we obtain the complex amplitude distribution at the output plane. Consequently, when $\varrho-\eta < r_0H_2$, i.e., when the point of observation is in the geometrically illuminated region of the output plane, $a(\varrho, d)$ may be regarded as the sum of the transmittance function and another term due to diffractional effects of the entrance pupil. In a similar way, when $\varrho-\eta > r_0H_2$, i.e., when the point of observation lies in the geometrical shadow, $a(\varrho, d)$ may be regarded as due to diffractional effects; that is, luminous points appear within the geometrical shadow.

For optical wavelengths (k sufficiently large) $kr_0 \gg 1$, so $v[\tau(d)] \gg 1$ when points of observation in the output plane are situated not far from the point $\eta[\tau(d)]$. For this reason, a good approximation to the Lommel functions is given by [13]

$$V_0(u, v) \simeq \sqrt{\frac{2}{\pi v}}\,\frac{\cos(v-\pi/4)}{1+(v/u)^2}, \tag{20a}$$

$$V_1(u, v) \simeq \sqrt{\frac{2}{\pi v}}\, \frac{v/u}{1+(v/u)^2} \cos(v-3\pi/4), \tag{20b}$$

$$U_1(u, v) \simeq \sqrt{\frac{2}{\pi v}}\, \frac{u/v}{1+(u/v)^2} \cos(v-3\pi/4), \tag{20c}$$

$$U_2(u, v) \simeq \sqrt{\frac{2}{\pi v}}\, \frac{(u/v)^2}{1+(u/v)^2} \cos(v+3\pi/4). \tag{20d}$$

The complex amplitude distribution at these points can be approximated by

$$a(\varrho, d) \simeq t(\varrho, d)+0\left(\frac{1}{\sqrt{v}}\right), \quad \text{for } \varrho-\eta < r_0H_2, \tag{21a}$$

$$a(\varrho, d) \simeq 0+0\left(\frac{1}{\sqrt{v}}\right), \quad \text{for } \varrho-\eta > r_0H_2 \tag{21b}$$

where $0(1/\sqrt{v})$ denotes the terms whose power is higher than or equal to $1/\sqrt{v}$.

From Eqs. (21) it follows that when $\varrho-\eta > r_0H_2$, $a(\varrho, d)\to 0$ no slower than $1/\sqrt{k\varrho}$, while when $\varrho-\eta < r_0H_2$, $a(\varrho, d)\to t(\varrho, d)$ as expected. Thus, when an entrance pupil of radious r_0 is situated on the plane $z = 0$ we have on plane $z = d$ an exit pupil of radius r given by the boundary of the geometrical shadow

$$u = v, \tag{22}$$

that is

$$r = r_0H_2[\tau(d)], \tag{23}$$

and centred at the point $(\eta_1[\tau(d)], \eta_2[\tau(d)])$ as shown in Fig. 1. Note that the presence of linear terms in the refractive index indicates that the exit pupil is off-axis.

From Eq. (23) it follows that the radius of the exit pupil depends on the output plane location. In other words, we characterize now the inhomogeneous medium by its effective transmittance function defined as

$$t_e(\varrho, d) = t(\varrho, d)\ \mathrm{cyl}\left(\frac{\varrho-\eta[\tau(d)]}{2r_0H_2[\tau(d)]}\right) \tag{24}$$

where cyl is the cylinder function centred at the point $\eta[\tau(d)]$ and $2r_0H_2[\tau(d)]$ is the diameter of the exit pupil [14].

For points in the neighborhood of the point $\eta[\tau(d)]$, $v[\tau(d)] \ll 1$, and the V_n function up to second-order terms in v can be approximated by:

$$V_0(u, v) \simeq \cos\frac{v^2}{2u}+0(v^4) \tag{25a}$$

$$V_1(u, v) \simeq \sin\frac{v^2}{2u}+0(v^4), \tag{25b}$$

and taking into account Eqs. (18a) and (10), the complex amplitude distribution is given by

$$a(\varrho, d) \simeq t(\varrho, d)\left\{1-\exp\left(i\,\frac{u[\tau(d)]}{2}\right)\right\}+0(v^4), \tag{26}$$

and the intensity can be written as

$$I(\varrho, d) \simeq I_0 \operatorname{sinc}^2\left(\frac{u[\tau(d)]}{4}\right) \tag{27}$$

where

$$I_0 = \frac{(kr_0)^2 n_0^3}{4H_1^2[\tau(\tilde{z}_p)]n_1(\tilde{z}_p)} \tag{28}$$

is the intensity at $u = v = 0$, that is, for points on the Fourier planes $z = \tilde{z}_p$, where p is an integer, which coincide with the centre of the spectrum, since $H_2(\tilde{z}_p) = 0$ is the Fourier transform condition [8].

For points $\varrho = \eta[\tau(z)]$, $v[\tau(z)] = 0$, and the two V_n functions entering Eq. (18a) reduce to

$$V_0(u, 0) = 1, \quad V_1(u, 0) = 0. \tag{29}$$

Hence the complex amplitude distribution is given by

$$a\left(\eta[\tau(z)], z\right) = t\left(\eta[\tau(z)], z\right)\left\{1-\exp\left(i\,\frac{u[\tau(z)]}{2}\right)\right\}. \tag{30}$$

Equation (30) is similar to Eq. (26) apart from terms in $0(v^4)$. Thus the intensity $I = |a|^2$ at these points is characterized exactly by the function $\operatorname{sinc}^2\left(u[\tau(z)]/4\right)$, and the zeros of intensity are given by

$$H_2[\tau(z)] = \frac{2m\lambda}{n_0 r_0^2} H_1[\tau(z)], \quad (m = \pm 1, \pm 2, \ldots,). \tag{31}$$

On the other hand, for image planes $z = z_p$, $H_1[\tau(z_p)] = 0$, so that $\dot{H}_2[\tau(z_p)] = 0$ [8]. Hence the intensity distribution becomes

$$I(\varrho, z_p) = |t(\varrho, z_p)|^2 \operatorname{cyl}\left(\frac{\varrho-\eta[\tau(z_p)]}{2r_0 H_2[\tau(z_p)]}\right) = |t_e(\varrho, z_p)|^2 \tag{32}$$

where the fact that cyl is a binary function of unit step (equal either to unity or to zero) has been used to replace cyl^2 by cyl.

From Equation (32) it follows that the effective transmittance function defined by Eq. (24) is valid and correct at every paraxial image plane. For Fourier planes $z = \tilde{z}_p$, $H_2[\tau(\tilde{z}_p)] = 0$ or $u[\tau(z_p)] = 0$, and taking into account Eqs. (16a), (16c) and (17), the intensity reduces to

$$I(\varrho, \tilde{z}_p) = I_0\left(\frac{2J_1\left(v[\tau(\tilde{z}_p)]\right)}{v[\tau(\tilde{z}_p)]}\right)^2 \tag{33}$$

where J_1 is the first-order Bessel function of the first kind. Eq. (33) is just the Airy pattern centred at the point $\eta[\tau(z_p)]$. Thus the first minimum (intensity zero) in the Fourier planes is given by

$$\varrho - \eta[\tau(\tilde{z}_p)] = \sqrt{(x - \eta_1[\tau(\tilde{z}_p)])^2 + (y - \eta_2[\tau(\tilde{z}_p)])^2} = 0.61\,\frac{\lambda H_1[\tau(\tilde{z}_p)]}{n_0 r_0}. \tag{34}$$

3. Conclusions

In this paper, we study the effect on the transmittance function in nonrotation-symmetric gradient-index material due to a circular pupil, and we characterize this material by its effective transmittance function defined as the transmittance function without pupil multiplied by the cylinder function. We also determine the intensity in image and transform planes. Therefore the complex amplitude distributions in the geometrically illuminated region and in the geometrical shadow have been evaluated in terms of the Lommel functions.

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