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On the accuracy of the stationary phase method

Grażyna Mulak

Institute of Physics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

The results of calculations of the amplitude of the wave diffracted at a straight edge, obtained from Van Kampen formulae, were compared with those obtained by using an approximation of Fresnel's integrals. The regions, where the first term of an asymptotic expansion describing the diffraction wave is satisfactory, were pointed. Some remarks concerning the influence of further terms of asymptotic expansion on Kirchhoff's integral evaluation were made.

1. Introduction

Many problems of optics reduce to finding Kirchhoff's integral over any surface, e.g., that of the lighting object, an exit pupil of an optical system, a hole in black screen or plane of the hologram. In the case, when the integration surface is large, this problem may, among others, be solved by using the asymptotic approach.

The substantial advantage arising from using this method is that Kirchhoff's integral can be evaluated by virtue of the disturbations originating from the several active points (critical points).

It seemed advisible to known, which are the quantitative relations between the asymptotic approximation and other results. To this end we have chosen Fresnel's diffraction at an infinite straight edge because of different reasons. One of them is that the problem is well described by Fresnel's integrals and the obtained results are in good agreement with the experimental data [1]. The second reason is that in our case, unlike in other forms of apertures, there is only one, isolated critical point of the second kind. This fact creates good conditions for analysis.

2. Method of Fresnel's integrals

The complex amplitude of the disturbance at observation point P is described by the formula [1]

$$U_P = B(C+iS) \tag{1}$$

where

$$B = -\frac{iA}{\lambda}\cos\delta\,\frac{\exp[ik(r'+s')]}{r'+s'}, \tag{1a}$$

$$C = b\left\{\left[\frac{1}{2}+\mathscr{C}(w)\right]-\left[\frac{1}{2}+\mathscr{S}(w)\right]\right\}, \tag{1b}$$

$$S = b\left\{\left[\frac{1}{2}+\mathscr{C}(w)\right]+\left[\frac{1}{2}+\mathscr{S}(w)\right]\right\}, \tag{1c}$$

$$b = \frac{\lambda}{2\left(\frac{1}{r'}+\frac{1}{s'}\right)\cos\delta}. \tag{1d}$$

$\mathscr{C}(w)$ and $\mathscr{S}(w)$ are Fresnel's integrals

$$\mathscr{C}(w) = \int_0^w \cos\left(\frac{\pi}{2}\tau^2\right)d\tau, \tag{2}$$

$$\mathscr{S}(w) = \int_0^w \sin\left(\frac{\pi}{2}\tau^2\right)d\tau,$$

where for diffraction at a straight edge

$$w = \sqrt{\frac{2}{\lambda}\left(\frac{1}{r'}+\frac{1}{s'}\right)}\,X\cos\delta, \tag{3}$$

λ – the wavelength. All the remaining denotations are shown in Fig. 1. The source point P_0 and the observation point P are located in (x, z) plane.

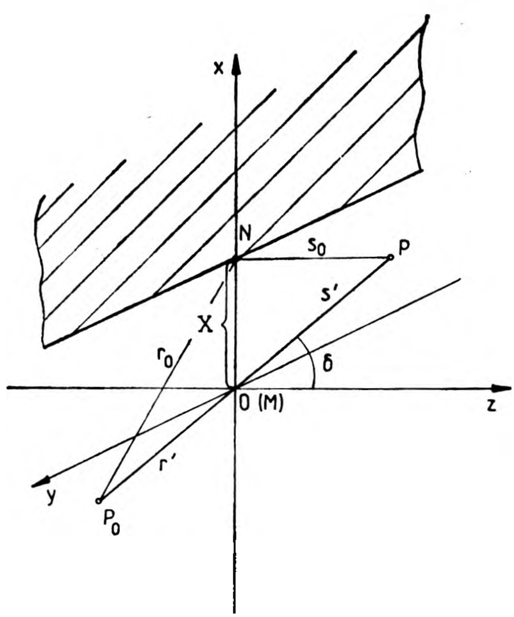

Fig. 1

The basic diffraction integral (1) may be rewritten in the form

$$U_P = \frac{A\exp[ik(r'+s')]}{r'+s'}\,\frac{(1+\mathscr{C}+\mathscr{S})-i(\mathscr{C}-\mathscr{S})}{2}. \tag{4}$$

For the sufficiently great values of w both integrals (2) may be approximated by [2]

$$\begin{aligned}\mathscr{C}(w) &\approx \frac{1}{2}+\frac{1}{\pi w}\sin\frac{\pi}{2}w^2+o\left(\frac{2}{\pi w^2}\right),\\ \mathscr{S}(w) &\approx \frac{1}{2}-\frac{1}{\pi w}\cos\frac{\pi}{2}w^2+o\left(\frac{2}{\pi w^2}\right).\end{aligned} \tag{5}$$

In this approximation the positions of extrema are preserved. The errors of the approximation calculated with respect to the data given in 4-digit tables [2] are specified in Table 1.

Table 1

w	$\mathscr{C}(w)$ exact., after [2]	$\mathscr{C}(w)$ approx., according to (5)	Error [%]
1	0.7799	0.8183	5
2	0.4883	0.5000	2.5
2.6	0.3889	0.3862	< 1
3	0.6057	0.6061	< 1
3.2	0.4663	0.4634	< 0.5
4.2	0.5417	0.5407	< 0.2
6	0.4995	0.5000	< 0.2
6.2	0.4676	0.4673	< 0.1
7.2	0.4887	0.4889	< 0.1

For w tending to ∞

$$U_P \to U_{P\infty} = \frac{A_{\exp}[ik(r'+s')]}{r'+s'}. \tag{6}$$

Let us introduce

$$\delta_F = \frac{U_P-U_{P\infty}}{U_{P\infty}}, \tag{7}$$

which will be convenient for further analysis.

In our case

$$\delta_F = \frac{(1+\mathscr{C}+\mathscr{S})-i(\mathscr{C}-\mathscr{S})}{2} -1. \tag{8}$$

And for great values of w

$$\delta_F = \frac{1}{\sqrt{2}\pi w}\left[-\cos\left(\frac{\pi}{4}+\frac{\pi}{2}w^2\right)-i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}w^2\right)\right]. \tag{9}$$

The amplitude and the phase are:

$$|\delta_F| = \frac{1}{\sqrt{2\pi w}}, \tag{9a}$$

$$a_F = \frac{5}{4}\pi + \frac{\pi}{2} w^2, \tag{9b}$$

respectively.

3. Method of the stationary phase

The complex amplitude in P may be presented as sum

$$U_P = U_P^{(g)} + U_P^{(d)}, \tag{10}$$

where $U_P^{(g)}$ – disturbance predicted by geometrical optics, called by Rubinowicz *the geometric-optics wave* [3],

$U_P^{(d)}$ – disturbance representing the diffraction effects, called by Rubinowicz *the diffraction wave* [3].

The stationary phase is appropriate for great values of wave number k. According to Van Kampen formulae [4] and taking into account only the first term of $U_P^{(d)}$, for sufficiently great k we have

$$U_P = \frac{\exp[ik(r'+s')]}{r'+s'} + \sqrt{\frac{\pi}{k|a_{02}|}}\,\frac{i\varepsilon_2 b_{00}}{a_{10}}\exp(ika_{00}). \tag{11}$$

In our case for a critical point of the second kind (point N in Fig. 1) we have

$$\begin{aligned}
a_{00} &= r_0 + s_0,\\
a_{01} &= 0,\\
a_{10} &= -\left(\frac{1}{r_0}+\frac{1}{s_0}\right)X,\\
a_{02} &= \frac{1}{2!}\left(\frac{1}{r_0}+\frac{1}{s_0}\right),\\
a_{20} &= \frac{1}{2!}\left(\frac{1}{r_0}+\frac{1}{s_0}\right) - X^2\left(\frac{1}{r_0^3}+\frac{1}{s_0^3}\right),\\
b_{00} &= -\frac{iA}{4\pi}\,\frac{r's_0+s'r_0}{(r_0 s_0)^2},\\
\varepsilon_2 &= e^{i(\pi/4)}.
\end{aligned} \tag{12}$$

The simplifying supposition $\cos\delta = 1$ was done. To obtain (12) the origin of the system of coordinates must be located at N.

For k tending to ∞ there remains the geometric-optics wave only

$$U_P \to U_{P\infty} = \frac{\exp[ik(r'+s')]}{r'+s}. \tag{13}$$

Analogically as in (7), using

$$\delta_C = \frac{U_P - U_{P\infty}}{U_{P\infty}}, \tag{14}$$

we obtain

$$\delta_C = \sqrt{\frac{\pi}{k|a_{02}|}}\,\frac{r'+s'}{A}\,\frac{b_{00}}{a_{10}}. \tag{15}$$

After substituting (12) into (11) we get

$$|\delta_C| = \frac{1}{2\sqrt{2\pi w}}\,D, \tag{15a}$$

where

$$D = \left(\frac{r'+s'}{r_0+s_0}\right)^{3/2} \frac{s_0 r' + s' r_0}{(r' s' r_0 s_0)^{1/2}};$$

and

$$a_C = \frac{5}{4}\pi + k[(r_0+s_0)-(r'+s')]. \tag{15b}$$

4. Discussion

As it follows from the comparison of formulae for δ (Eqs. (9) and (15)), the full agreement of δ_F and δ_C takes place when geometrical factor D in Eq. (15a) equals 2. This factor depends upon the ratios (r'/X) and (s'/X) exclusively and its greatest value 2 is reached when $X = 0$. But $X = 0$ implies $w = 0$. In this region, because of small value of w, the comparison of formulae (9) and (15a) cannot be made. However, already for $w = 1$ (Tab. 1) the comparison may be done. The above considerations are illustrated in Fig. 2. For the given position of source $(r'/X) = 1$ and for various positions of the observation point represented by a pencil of straight lines $(s'/X) = \text{const.}$ there were marked the corresponding values of D. Assuming that $(\lambda/X) = 5\times10^{-5}$ we have drown family of lines $w = \text{const}$ (isophotes).

As seen from this figure, the stationary phase method gives the value of amplitude, predicted by geometrical optics, quicker (nearer the shadow boundary) than the Fresnel's integrals method. An agreement between a_F and a_C

occurs in the regions, where the following approximation can be made:

$$r_0 \approx r' + \frac{1}{2}\frac{X^2}{r'},$$
$$s_0 \approx s' + \frac{1}{2}\frac{X^2}{s'}. \tag{16}$$

Then

$$k[(r_0+s_0)-(r'+s')] \approx \frac{\pi}{2} w^2, \text{ and } a_C \approx a_F.$$

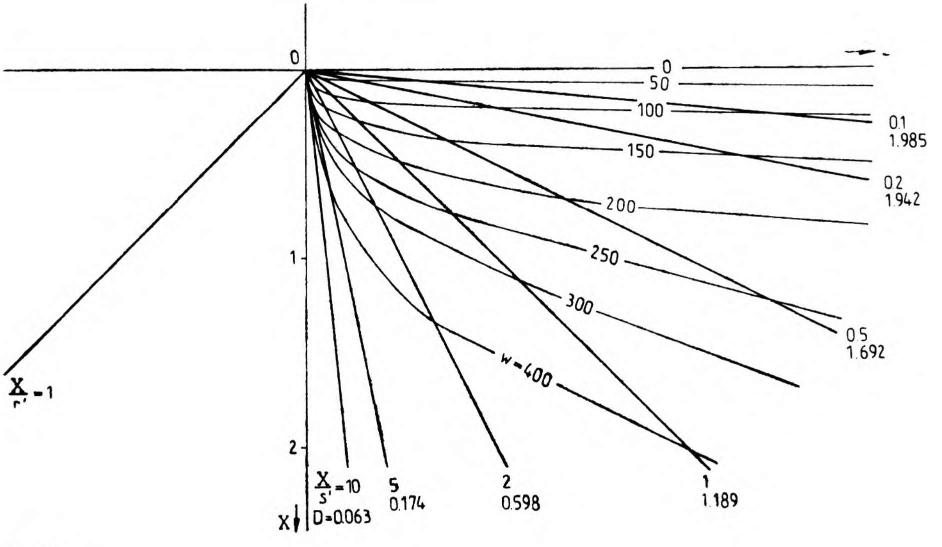

Fig. 2

This approximation holds when the first from the neglected terms of the binomial expansion satisfies the conditions

$$k\frac{1}{8}X^4\left(\frac{1}{r'^3}+\frac{1}{s'^3}\right) \ll 2\pi, \tag{17}$$

it is, when the assumption of Fresnel diffraction is fulfilled. If we assume that this term is equal to $2\pi/100$, then X, D, and w for the given positions r' and s' take the values presented in Table 2. It can be seen from this table that the ratio

Table 2

r' [cm]	s' [cm]	X [cm]	w	D	
1	1	3.8×10^{-2}	10.6	1.9979	
1	2	4.3×10^{-2}	10.6	1.9986	
1	0.5	2.5×10^{-2}	8.7	1.9982	$\lambda = 5\times10^{-5}$ cm
2	2	6.3×10^{-2}	12.6	1.9985	
0.5	0.5	2.4×10^{-4}	9.6	1.9965	

$(|\delta_C|/|\delta_F|) = D/2$ differs from 1 by about 1‰. Thus, in the region of Fresnel approximations both the descriptions are in perfect agreement.

The question arises, what is the contribution of further terms of expansions describing the effect of the critical points on the diffraction wave. For the critical points of first kind (point M in Fig. 1) we have [4]

$$U_M(P) = \frac{\pi e^{ika_{00}}}{\sqrt{|a_{20}a_{02}|}}\,\varepsilon_1\varepsilon_2\left[b_{00} + \frac{ib_{20}}{2ka_{20}} + \frac{ib_{02}}{2ka_{02}} - \ldots\right] \tag{18}$$

where the first term describes the geometric-optics wave, and the subsequent ones are the contributions to the diffraction wave like those

$$U_N(P) = e^{ika_{00}}\sqrt{\frac{\pi}{k|a_{02}|}}\,\frac{i\varepsilon_2}{a_{10}}\left[b_{00} - \frac{b_{10}}{ika_{10}} + \frac{ib_{02}}{2ka_{02}} + \ldots\right] \tag{19}$$

originating from the second kind critical point N on the diffracting edge. In our case the estimation of the ratios of the second and third terms to the first term in expressions (18) and (19) yields:

– for M point

$$\frac{b_{20}}{2ka_{20}b_{00}} = \frac{b_{02}}{2ka_{02}b_{00}} = \frac{3}{4k}\,\frac{r'^2+s'^2}{r's'(r'+s')}, \tag{20}$$

– for N point

$$\begin{aligned}\frac{b_{10}}{ka_{10}b_{00}} &= -\frac{1}{k}\,\frac{r'r_0^2s_0+s's_0^2r_0+2(r's_0^3+s'r_0^3)}{r_0s_0(r_0+s_0)(r's_0+s'r_0)},\\ \frac{b_{02}}{2ka_{02}b_{00}} &= -\frac{1}{2k}\,\frac{r'r_0^2s_0+s's_0^2r_0+2(r's_0^3+s'r_0^3)}{r_0s_0(r_0+s_0)(r's_0+s'r_0)}.\end{aligned} \tag{21}$$

Putting $r' = s' = a$, $r_0 = s_0 = b$, we get jointly:

– for M point

$$\frac{3}{4\pi}\,\frac{\lambda}{a},$$

– for N point (22)

$$-\frac{9}{8\pi}\,\frac{\lambda}{b}.$$

As seen from (22), the substantial influence of further terms of $U_M(P)$ will be marked in the vicinity of the aperture plane, while those of $U_N(P)$ will be seen in the nearest vicinity of the diffracting edge.

Figure 3 shows the comparison of the results of both the methods in the vicinity of the shadow boundary. The intensities were calculated from Eqs. (11) and (4). In the case of small values of w we substitute in Eq. (4) quickly

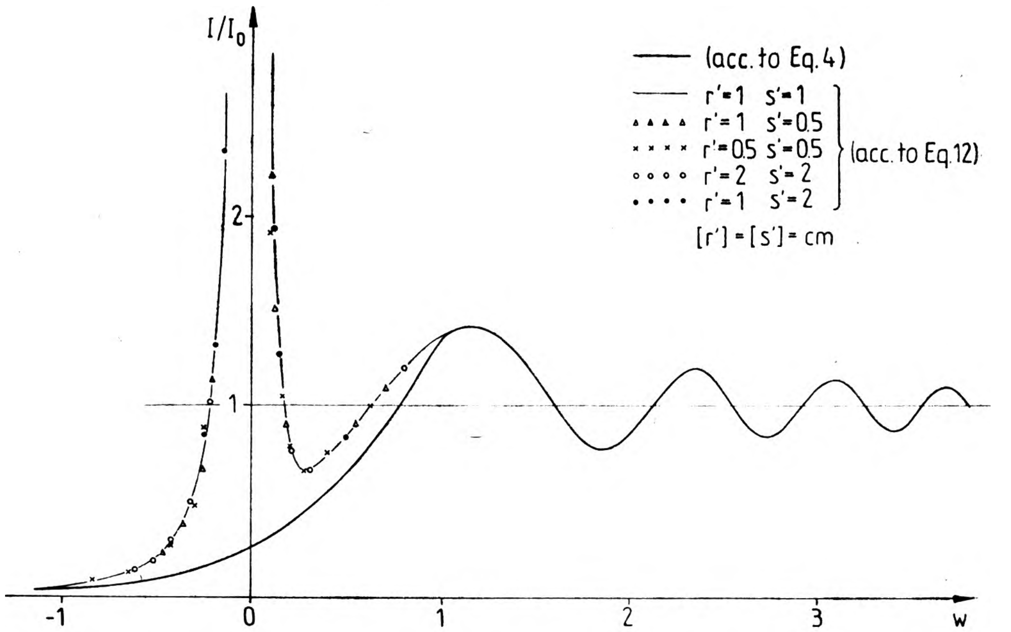

Fig. 3

convergent series [1]

$$\mathscr{C}(w) = w\left[1-\frac{1}{2!5}\left(\frac{\pi}{2}w^2\right)^2+\frac{1}{4!9}\left(\frac{\pi}{2}w^2\right)^4-\ldots\right],$$
$$\mathscr{S}(w) = w\left[\frac{1}{1!3}\left(\frac{\pi}{2}w^2\right)-\frac{1}{3!7}\left(\frac{\pi}{2}w^2\right)^3+\ldots\right]. \tag{23}$$

The discontinuity introduced by the division of the disturbation into $U^{(g)}$ and $U^{(d)}$ is visible in this Figure. In the shadow region $U^{(g)}$ disappears and $U^{(d)}$ at $w = 0$ is also discontinuous. In this case the singularity caused by coincidence of critical points of the first and the second kinds requires a special treatment.

5. Conclusions

The results of Fresnel approximations, according to [1], are in agreement with the experimental data. While deriving Fresnel's formulae it has been assumed that the sizes of the domain of integration are small with respect to the distances r' and s' [4, 5]. This agreement is strange considering that the assumptions mentioned above are not fully satisfied in the case of straight edge.

In the stationary phase method such assumptions are not required. The requirement (17) is in general not necessary [5]. For the distances violating (17) and for great values of k, the oscillations of the quadratic phase factor will be so rapid, that the contribution to the Kirchhoff's integral arise only from the critical points [4, 5], where the rate of changes of phase is minimum.

It is obvious that the Fresnel's approximation does not hold in the near field region for a strong angular divergence of beams and in the *intermediate region* (i.e., between the near and far fields). Both the descriptions give the oscillations of intensity about the value predicted by the geometrical optics. It seems, however, that the description of the field in the intermediate region obtained by the stationary phase method may be closer to reality than the Fresnel's one. This supposition may be best verified by experiment.

The above results permit us to examine the holographic imaging by means of the critical point methods. Under limit resolution conditions, when all the sources taking part in the imaging area at distances comparable with the hologram size, the region of the shadow will be much more complex and a detailed information about the description of the field formed by each source is required.

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К вопросу точности метода стационарной фазы

Были сравнены результаты вычислений амплитуды волны, дифрагированной на прямолинейном крае, полученных по формулам Ван Кампена и с применением интегралов Френеля. Области, где первые члены асимптотических развитий удовлетворенно описывают волну дифракции, были показаны. Были произведены некоторые примечания относительно влияния высших членов асимптотических развитий на оценку интеграла Кирхгофа.