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Observability of nonlinear waves in optical fibers

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Stability of plane and stationary nonlinear waves in optical fibers is discussed as solutions of the exponential nonlinear Schrödinger equation.

1. Introduction

In recent years, there has been a growing interest in the nonlinear wave theory in optical fibers [1]–[4]. This interest has been stimulated by the invention of powerful mathematical tools for analysing the nonlinear wave phenomena. From the experimental point of view, the main question is the observability of nonlinear waves in the optical fibers, hence the importance of the theoretical problem of stability of nonlinear waves and especially, the stability of solitons is a very important subject to study.

It is well known that the fundamental equation describing the nonlinear evolution of pulse envelope, as the pulse propagates along the fiber, is the nonlinear Schrödinger equation (NLSE) or its generalizations [5]–[10]. These equations have been derived basing on the small amplitude assumption by introducing a small nonlinear parameter ε ($|\varepsilon| \ll 1$) in the following expansion:

$$u = u_0 + \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-\infty}^{\infty} u_m^{(n)}(\xi,\tau) e^{im(kx-\omega t)} \tag{1}$$

where: $u_m^{(1)} = 0$, $m \neq \pm 1$, $u_m^{(1)*} = u_{-m}^{(1)}$, $\tau = \varepsilon(t-\lambda x)$, $\xi = \varepsilon^2 x$. The derived NLSE has the following form [5], [6]:

$$iu_{1\xi}^{(1)} + \beta |u_1^{(1)}|^2 u_1^{(1)} + \alpha u_{1\tau\tau}^{(1)} = 0. \tag{2}$$

Here β and α are the nonlinear and dispersive coefficients, respectively.

We now proceed to consider the exponential NLSE introduced recently to nonlinear plasma physics by D'Evelyn and Morales [11], Kaw et al. [12] and Sheerin and Ong [13]

$$iu_{\xi} + \beta(1-e^{-|u|^2})u + \alpha u_{\tau\tau} = 0. \tag{3}$$

In nonlinear optics, this equation does not include the Raman nonlinear dissipation term [6] which may become important for the case of strong electric field E.

We simply neglect this term to make the equation analytically tractable, though we are aware that this makes our approach only approximate. The full problem is left for future numerical studies.

For derivation of Equation (3), no assumption of small wave amplitude was made. However, in this limit, the nonlinear term of Eq. (3) transforms to that of Eq. (2). Moreover, for relatively small amplitudes of the incident pulse the nonlinear effects may be neglected and we actually have the linear regime. Similarly, for infinitely large amplitudes, the waves interact weakly with the medium and thus the nonlinear term in the model equation may be replaced by a linear one. Thus, the saturated exponential nonlinearity effect plays an important role in maintaining waves of finite amplitudes. It is worth noting that Eqs. (2) and (3) are equivalent at certain limits. It has been shown by D'EVELYN and MORALES [11] that both Eqs. (2) and (3) give essentially the same results for normalized amplitudes smaller than 0.2. For amplitudes greater than 0.5, the influence of the exponential nonlinearity causes a significant difference from the estimation predicted by the cubic nonlinearity.

This paper is organized as follows. The next Section presents the derivation of the exponential nonlinear Schrödinger equation (henceforth, ENLSE) assuming exponential dependence of the refractive index on the electric field. Section 3 shows a necessary condition for the modulational stability of the ENLSE. In Section 4, the Infeld–Rowlands method [15] is applied for this equation to study a stability of waves as its solutions. Numerical calculations and results are presented in Sections 5 and 6, respectively. The final part is a short summary.

2. Derivation of model equation

The refractive index of the medium can be represented in the form of a power expansion of the electric field. See KODAMA [5], KARPMAN [14], and references therein. There are several effects which may lead to such dependence. One of them is the Kerr effect which arises from the orientation of anisotropic molecules in the electric field [14]. Also electrostriction and the ionization of the medium by the incident electric field may be the causes [14].

Let us express the refractive index in the following form:

$$n = \sum_{l=0}^{\infty} n_l |E|^{2l}.$$

Here, we used the even powers of the electric field because they come from the interaction of dipoles, quadrupoles, and so on, with the incident electric field. This expression should be valid for arbitrary value of E, and in the limit of infinite field E the refractive index should be finite. It suggests that the reasonable choice is

$$n = n_l + n_2[1 - \exp(-|E|^2)].$$

The linearly polarized optical wave pulse in an optical fiber is given by the equation

$$E_{xx}-\frac{1}{c^2}D_{l_{tt}}=\frac{2n_2n_0}{c^2}[(1-e^{-|E|^2})E]_{tt} \tag{4}$$

where n_2 represents a nonlinear part of the refractive index,

$$n=n_l+n_2(1-e^{-|E|^2}). \tag{5}$$

The optical Kerr effect arises from the orientation of anisotropic molecules in the wave field E. A good reference is PIEKARA [16] who used the following dependence of the refractive index n of liquids on the intensity of electric field E:

$$n=n_l+n_2|E|^2+n_4|E|^4+n_6|E|^6.$$

The sequence of terms such as $|E|^2$, $|E|^4$, and so on, in this formula is a consequence of an interaction of the electric field E with dipoles, quadrupoles, and so on, respectively. In the limit $|E|^2-0$, for weak incident electric field, it is quite sufficient to drop higher-order terms to obtain the classical quadratic dependence of the refractive index on the electric field envelope. For stronger fields, however, such approach is very rough and a natural correction is to use all three terms and even to extend it by taking the next ones.

So, here we generalized quadratic dependence of the refractive index n on the electric field E to the exponential one. Let us notice that for $|E|\ll 1$ the last term in the formula (5) transforms to $n_2|E|^2$.

In expansion (5), n_l is a linear part of the refractive index and n_2 is a small constant. We assume that

$$n_0=n_l(\omega_0), \tag{6a}$$

and D_l is a linear part of the displacement vector

$$D_l=\int_{-\infty}^{t} n_l^2(t-t')E(t')dt' \tag{6b}$$

where we extract the slowly varying complex envelope $u(x,t)$ of the short-wavelength optical field,

$$E(x,t)=u(x,t)e^{i(qx-\omega_0 t)}. \tag{7}$$

Here q is the propagation constant.

Substitution of expression (7) into Eq. (4) leads to the equation

$$\{\partial_x^2+2iq\partial_x-q^2+k_0'^2+2ik_0k_0'\partial_t-[(k_0')^2+k_0k_0'']\partial_t^2\}u$$

$$=\frac{2n_2n_0}{c^2}e^{i\omega_0 t}\partial_t^2[(1-e^{-|u|^2})ue^{-i\omega_0 t}] \tag{8}$$

where: $k^2=\dfrac{\omega^2n_l^2}{c^2}$, $k_0^2=\dfrac{\omega_0^2n_0^2}{c^2}$, $k_0'=\left.\dfrac{\partial k}{\partial\omega}\right|_{\omega_0}$, $k_0''=\left.\dfrac{\partial^2 k}{\partial\omega^2}\right|_{\omega_0}$,

and ∂ means the partial derivatives operator. For samples short enough the r.h.s. of (8) may be replaced by

$$-2\frac{n_2k_0^2}{n_0}(1-e^{-|u|^2})u. \tag{9}$$

We also use the slowly varying envelope approximation

$$k_0 \gg (\partial_x - 2k_0'\partial_t). \tag{10}$$

Finally, we get from (8)

$$i(u_x+k_0'u_t)-\frac{1}{2}k_0''u_{tt}+\frac{\omega_0 n_2}{c}(1-e^{-|u|^2})u = 0. \tag{11}$$

Using a stretched variable

$$\tau = t-k_0'x, \tag{12}$$

we obtain

$$iu_x+\alpha u_{\tau\tau}+\beta(1-e^{-|u|^2})u = 0 \tag{13}$$

where:

$$\alpha = -\frac{1}{2}k_0'', \ \beta = \frac{\omega_0 n_2}{c}. \tag{14}$$

For small amplitudes u, $1-e^{-|u|^2}$ reduces to $|u|^2$, and we have the NLSE.

3. Modulational stability

Equation (13) has a plane-wave solution

$$u = u_0\exp(-i\Delta kx) \tag{15}$$

where u_0 is an arbitrary constant, and

$$\Delta k = -\beta(1-\exp(-u_0^2)). \tag{16}$$

We perturb the wave by the small amplitude disturbance $\delta u(x,\tau)$

$$u(x,\tau) = (u_0+\delta u)\exp(-i\Delta kx). \tag{17}$$

Substitution of the above formula into Eq. (13) and removal of the nonlinear terms leads to the following equation for the disturbance δu

$$i\delta u_x+\alpha\delta u_{\tau\tau}+\beta\exp(-u_0^2)[u_0^2(\delta u+\delta u^*)] = 0. \tag{18}$$

Setting $\delta u = \bar{u}+i\bar{v}$ and separating the real and imaginary parts, we obtain

$$\alpha\bar{u}_{\tau\tau}-\bar{v}_x+2\beta u_0^2\bar{u}\exp(-u_0^2) = 0, \tag{19a}$$

$$\alpha\bar{v}_{\tau\tau}+\bar{u}_x = 0. \tag{19b}$$

Looking for solutions of the form

$$\bar{u}, \bar{v} \sim \exp[i(Kx-\Omega\tau)], \tag{20}$$

we find dispersion relation

$$K^2 = \alpha\Omega^2[\alpha\Omega^2 - 2\beta u_0^2 \exp(-u_0^2)], \tag{21}$$

which has the solution

$$K = \pm[\alpha\Omega^2(\alpha\Omega^2 - 2\beta u_0^2 \exp(-u_0^2)]^{1/2}. \tag{22}$$

This equation admits of an oscillatory instability for $\alpha > 0$ and

$$\Omega^2 \leqslant \Omega_c^2 \equiv \frac{2\beta u_0^2 \exp(-u_0^2)}{\alpha}. \tag{23}$$

A maximum growth rate γ corresponds to the maximum of an imaginary part of K and occurs for

$$\Omega = \Omega_c/\sqrt{2}. \tag{24}$$

Its value may be easily calculated from (22)

$$\max|\mathrm{Im}K| = |\alpha|\frac{\Omega_c^2}{2}. \tag{25}$$

The modulational instability is a very important subject to study in nonlinear fiber optics [9], [10] and in other areas of science [17]–[19]. It relays on a process in which small amplitude perturbations from the steady-state grow exponentially as a result of an interaction between Fourier modes. From Eq. (23) we see that a critical frequency Ω_c, also called the cut-off, depends both on the value of the amplitude u_0 and the quotient β/α.

4. Stabilities of the exponential nonlinear Schrödinger stationary waves

This part of the paper presents results of the stability analysis of nonlinear waves, solitons and shock waves like solutions of ENLSE, which is rewritten here in the form

$$iu_x + u_{tt} + \beta(1 - e^{-|u|^2})u = 0. \tag{26}$$

This equation is obtained from (13) by the transformation $\tau \to \sqrt{|\alpha|}t$, where α is positive. In the case of a negative α, Eq. (26) may be also derived both by the transformation of t and the renaming of the nonlinear coefficient β and the coordinate x to $-\beta$ and $-x$, respectively.

4.1. Stationary wave solutions

We now look for stationary envelope solutions

$$u = u_0(\tau)e^{i(ct/2 + bx)} \tag{27}$$

where $\tau = t - cx$. Equation (26) leads to

$$u_{0_{\tau\tau}} - pu_0 - \beta u_0 e^{-u_0^2} = 0 \tag{28}$$

where we defined

$$p \equiv b + \frac{c^2}{4} - \beta.$$

After integration of Eq. (28) multiplied by $u_{0\tau}$, we get

$$u_{0\tau}^2 = pu_0^2 - \beta e^{-u_0^2} + pl \equiv Y(u_0). \tag{29}$$

Here l is another integration constant.

The qualitative nature of the solution of the ENLSE may be determined from consideration of the function $Y(u_0)$ which should be bounded for bounded u_0 and must possess double roots. This happens when

$$Y'(u_0) = Y(u_0) = 0.$$

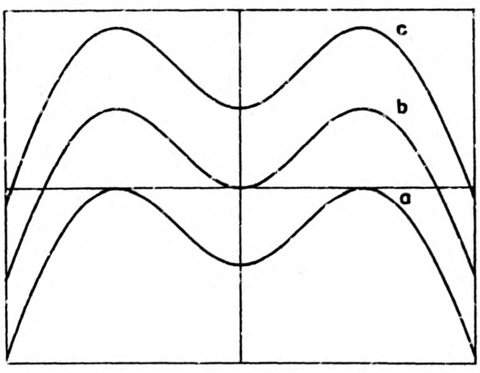
Hence, we find the condition for β and p

$$p\beta < 0, \; p(\beta + p) \leqslant 0 \tag{30}$$

and values of l corresponding to the double roots

$$l_{\min, \max} = \min, \max \left\{ \frac{\beta}{p}, \ln\left(-\frac{p}{\beta}\right) - 1 \right\}. \tag{31}$$

We now consider the case of $\beta > 0$ assuming that the conditions (30) are satisfied. We call this case the "soliton" case. For $l = l_{\max}$ and $l = l_{\min}$, we have linear waves and solitons as solutions of Eq. (29). For $l_{\max} > l > l_{\min}$ and $l < l_{\min}$ there are periodic waves. The general behaviour of the function $Y(u_0)$ with u_0 and $\beta > 0$ is shown schematically in Fig. 1. The other case we shall take into consideration is for $\beta > 0$,



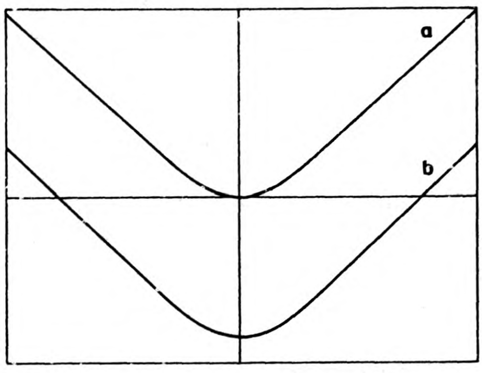


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Fig. 1. Phase diagrams for the exponential nonlinear Schrödinger equation for the case $\beta = 1$: a — linear wave limit, b — soliton, c — cnoidal wave

Fig. 2. As for Fig. 1, but here: a — linear wave limit, b — cnoidal wave

$$p < 0,\ p(\beta+p) > 0. \tag{32}$$

For $l > \beta/p$, there exists a range of periodic waves as solutions of Eq. (29). For $l = \beta/p$ there is a linear wave. See Figure 2 for a qualitative behaviour of the function $Y(u_0)$.

The case of the negative value of β ($\beta < 0$) is applicable in the optical fiber context when the dispersive coefficient $\alpha < 0$. See the comment below formula (29). If conditions (30) are satisfied we have a range of periodic waves for $l_{min} < l < l_{max}$. For $l = l_{min}$ and $l = l_{max}$, there are the linear wave and shock wave, respectively. We call this case the "shock-wave" case. See Fig. 3.

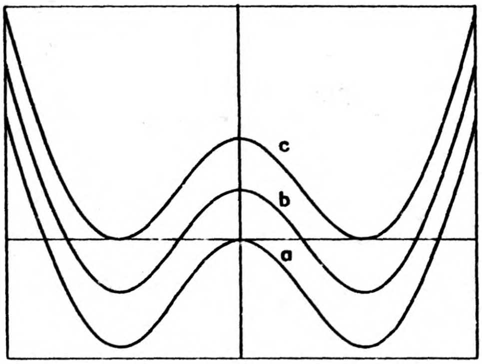


Fig. 3. Case of $\beta = -1$: a — linear wave limit, b — cnoidal wave, c — shock wave

4.2. Stability analysis

One approach to the study of the stability of optical pulses as solutions of a nonlinear wave equation is to assume a small amplitude and long period perturbations and consider whether or not this perturbation grows with distance. This approach allows us to approximate a nonlinear wave equation for perturbation by a linear equation: we superimpose a small disturbance of envelope with a long period and small amplitude upon the steady state given by Eq. (29)

$$u = [u_0(\tau)+\delta u_1(\tau)e^{i(\omega\tau+kx)}+\delta u_2(\tau)e^{-i(\omega\tau+k^*x)}]e^{i(c\tau/2+bx)}. \tag{33}$$

Here we have introduced coordinates of the moving frame

$$\tau = t-cx,\ x = x. \tag{34}$$

Physically, it means that in the moving frame the nonlinear wave does not change its form with the distance, whereas the disturbances depend on the distance.

In the coordinates of the moving frame, the ENLSE takes the form

$$i(u_x-cu_\tau)+u_{\tau\tau}+\beta(1-e^{-|u|^2})u = 0. \tag{35}$$

Substituting (33) and dropping nonlinear terms, we find

$$\hat{L}\delta u_+ - k\delta u_- + 2i\omega\delta u_{+\tau} - \omega^2\delta u_+ = 0, \tag{36a}$$

$$L\delta u_- - k\delta u_+ + 2i\omega\delta u_{-\tau} - \omega^2\delta u_- = 0 \tag{36b}$$

where the following notation is used:

$$L = \partial_\tau^2 - p - \beta e^{-u_0^2},$$
$$\hat{L} = L + 2\beta u_0^2 e^{-u_0^2},$$
$$\delta u_\pm = \delta u_1 \pm \delta u_2^*,$$

and the astersk denotes the complex conjugate. In further calculations we assume ω to be small and use the following expansion:

$$k = k_1\omega + k_2\omega^2 + \ldots, \tag{37a}$$

$$\delta u_+ = \delta u_{+0} + \omega\delta u_{+1} + \ldots, \tag{37b}$$

$$\delta u_- = K(\delta u_{-0} + \omega\delta u_{-1} + \ldots). \tag{37c}$$

Here K is an arbitrary constant which will be determined in the future. From the zeroth- and first-order equations in ω after an elimination of secular terms, we obtain

$$\delta u_{-0} = u_0, \tag{38a}$$

$$\delta u_{+0} = u_{0\tau}, \tag{38b}$$

$$\delta u_{-1} = u_0 + \frac{2iK - k_1}{2\eta K} P_0, \tag{38c}$$

$$\delta u_{+1} = u_{0\tau} + \frac{i}{2\bar{\beta}}(2 + ik_1 K\varkappa) Q_0 + \frac{k_1 K}{2} Q_2, \tag{38d}$$

where:

$$u_0 \int \frac{d\tau}{u_0^2} = \eta\tau u_0 + P_0(\tau), \tag{39a}$$

$$u_{0\tau} \int \frac{d\tau}{u_{0\tau}^2} = \bar{\beta}\tau u_{0\tau} + Q_0(\tau), \tag{39b}$$

$$u_{0\tau} \int \frac{u_0^2 d\tau}{u_{0\tau}^2} = \varkappa\tau u_{0\tau} + Q_2(\tau). \tag{39c}$$

Here, P_0, Q_0 and Q_2 are periodic functions with the same period as the nonlinear wave λ.

In the second-order of ω, we find

$$\hat{L}\delta u_{+2} - k_1 K\delta u_{-1} - k_2 K u_0 + 2i\delta u_{+1\tau} - u_{0\tau} = 0, \tag{40a}$$

$$L\delta u_{-2} - \frac{k_1}{K}\delta u_{+1} - \frac{k_2}{K} u_{0\tau} + 2i\delta u_{-1\tau} - u_0 = 0. \tag{40b}$$

We use the following properties of the operators L and $\hat{L}$:

$$Lu_0 = 0, \tag{41a}$$

$$\hat{L}u_{0\tau} = 0, \tag{41b}$$

and the fact that they are self-adjoint. Multiplying (40a) by $u_{0\tau}$ and integrating over the period λ, we get

$$2i\langle u_{0\tau}\delta u_{+1\tau}\rangle - k_1 K\langle u_{0\tau}\delta u_{-1}\rangle - \langle u_{0\tau}^2\rangle = 0, \tag{42}$$

where we used the definition

$$\langle f\rangle = \frac{1}{\lambda}\int_0^\lambda f d\tau.$$

Similarly, multiplying Equation (40b) by u_0 and integrating, we find

$$2i\langle u_0\delta u_{-1\tau}\rangle - \frac{k_1}{K}\langle u_0\delta u_{+1}\rangle - \langle u_0^2\rangle = 0. \tag{43}$$

The above Equations (38c), (38d), (42) and (43) are indispensable to obtain a dispersion relation. After the straightforward but lengthy calculation we find

$$AO_1 BO_1 k_1^4 + (AO_2 BO_1 + AO_1 BO_2 + AO_3 BO_3)k_1^2 + AO_2 BO_2 = 0, \tag{44}$$

where:

$$AO_1 = \bar{\beta}\langle u_{0\tau}P_0\rangle,$$
$$AO_2 = -2\eta(2\langle u_{0\tau}Q_{0\tau}\rangle + \bar{\beta}\langle u_{0\tau}^2\rangle),$$
$$AO_3 = 2(\bar{\beta}\langle u_{0\tau}P_0\rangle + \varkappa\eta\langle u_{0\tau}Q_{0\tau}\rangle - \eta\bar{\beta}\langle u_{0\tau}Q_{2\tau}\rangle),$$
$$BO_1 = \eta(\varkappa\langle u_0 Q_0\rangle - \bar{\beta}\langle u_0 Q_2\rangle),$$
$$BO_2 = 2\bar{\beta}(2\langle u_{0\tau}P_0\rangle - \eta\langle u_0^2\rangle),$$
$$BO_3 = 2(\eta\langle u_0 Q_0\rangle - \bar{\beta}\langle u_{0\tau}P_0\rangle).$$

Equation (44) is a very general test for stability of nonlinear waves. This equation usually gives complex k_1 for real values of the coefficients. If $\mathrm{Im}(k_1) < 0$, the effective amplitude of the wave will grow boundlessly. This effect is called instability. For fourth-order polynomial such as (44), if $\mathrm{Im}(k_1) > 0$, another mode exists for which $\mathrm{Im}(k_1) < 0$. So, both damping and blowing modes may exist.

To calculate the roots of Equation (44), we must find out numerically the number of quantities such as β, $\langle u_0\rangle$, $\langle u_0^2\rangle$, $\langle\exp(-u_0^2)\rangle$, $\langle u_0^{-2}\rangle$. For this purpose, we have used the Gauss-quadrature method for an integration. Other quantities are expressible by the above mentioned ones and may be calculated analytically. We do not present explicitly the lengthy formulae, however. They may be found from Eqs. (28), (29) and (39). For details of such calculations for other equations see, *e.g.*, [20], [21].

Changing the parameter l, we change amplitude of wave and by that way we can pass through the whole range of nonlinear waves which are solutions of the ENLSE equation. Our method is now well established and may be summarized as follows. We choose l just a bit bigger than the smallest one and solve the dispersion relation (44). If there is complex part of k, we claim that the wave (for certain l) is unstable. Repeated calculations allowed us to draw k vs. l.

5. Numerical calculations

We now describe the numerical calculations performed for solving the dispersion relation (44). We have applied the Gauss-quadrature procedure from CERN library to calculate integrals, such as

$$\langle u_0 \rangle \equiv \frac{1}{\lambda}\int_0^{\lambda} u_0 d\tau \equiv \frac{1}{b-a}\int_a^b \frac{u_0 du_0}{u_{0\tau}} \tag{45}$$

where a and b are the roots of equation $u_{0\tau}^2 = 0$. This procedure has been tested for analytical integrals. Besides, $\langle u_0 \rangle$ should be exactly equal to zero for the case of positive β and for $l < l_{min}$ (Fig. 1) and for $l > l_{max}$ (Fig. 2). Our method has given values approximately equal to 10^{-4} at 500 main grid points of the integration region. An accuracy 10^{-5} has required a double computer time. See Fig. 4 for the

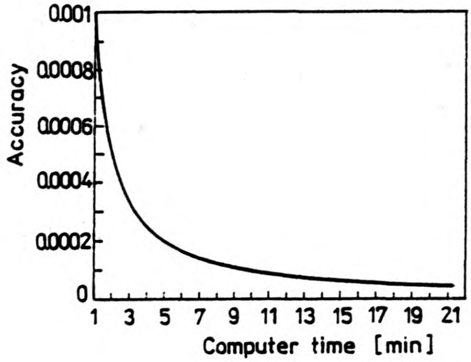


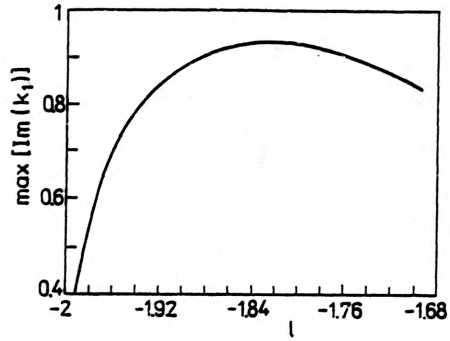
Fig. 4. Dependence of the accuracy of the numerical calculations on a computer time

dependence of the accuracy on a computer time. Additionally, the regions both at b and a (see Eq. (45)) have been divided to 500 mesh points. In order to verify this accuracy, we have performed a standard numerical test doubling the number of divisions of the integration region. No significant changes have occurred in our results.

We have also applied the procedure RZERO from CERN library to calculate the roots of equation $u_{0\tau}^2 = 0$. The accuracy of performed calculations has been 10^{-6}. Other quantities have been calculated on the basis of analytical formulae. All calculations have been made on a PC/AT computer. Every step for a parameter l required about 16 min. of CPU time at 500 divisions of the integration region. The numerical calculations have been carried out in double precision.

6. Numerical results

In this part of the paper, we present numerical results for the ENLSE comparing them, whenever possible, with those for the NLSE obtained by INFELD and ROWLANDS [15]. Firstly, we discuss the soliton case. See Section 4 for the meaning. We have a range of periodic waves for $l_{min} < l < l_{max}$ and $l < l_{min}$ and the soliton for



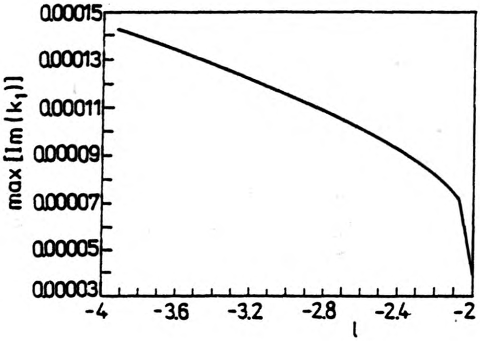


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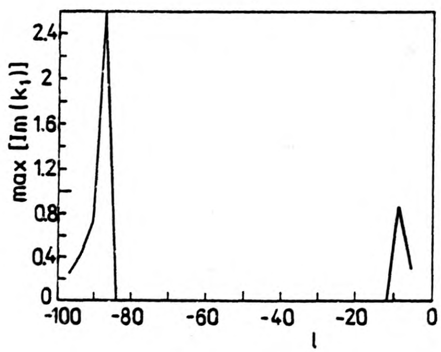
Fig. 5. Maximum value of imaginary part of the dispersion relation solution for $\beta = 1$, $b = 0.25$, $c = 1$ and for $l_{max} < l < l_{min}$. The smallest value of l corresponds to the soliton. A case of small amplitudes

Fig. 6. As for Fig. 5, but $l < l_{max}$. The largest value of l corresponds to the soliton

$l = l_{min}$. It has been found that for small amplitude waves, all periodic waves are unstable, although a weak instability exists for $l < l_{min}$. The soliton is stable to the perturbations. See Figs. 5 and 6. These results agree with those for the NLSE, which is valid for small amplitude limit, except the region $l < l_{min}$, where stability has been found [15]. The explanation of this apparent discrepancy is inherent in different physical meanings of these equations. Here, both small saturated nonlinear effects have been applied and possibly numerical inaccuracies are involved which caused a very small (10^{-4}) growth rate and thus instability. We have made sure that qualitative nature of the stability for $l < l_{min}$ does not depend on the value of the parameter p/β, connected with a wave amplitude. For larger amplitude waves in the region of $l_{min} < l < l_{max}$, the instability region becomes narrower and we have found only instabilities both at the soliton and the linear wave. See Figs. 7 and 8. This behaviour may be explained in the following way. Small amplitude waves have too



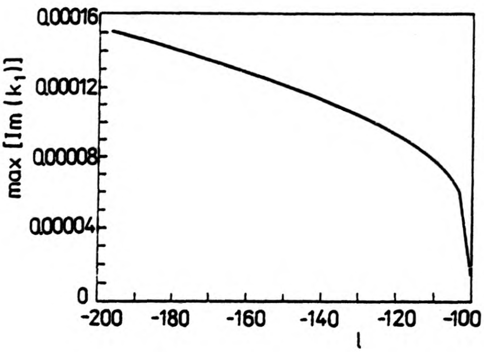


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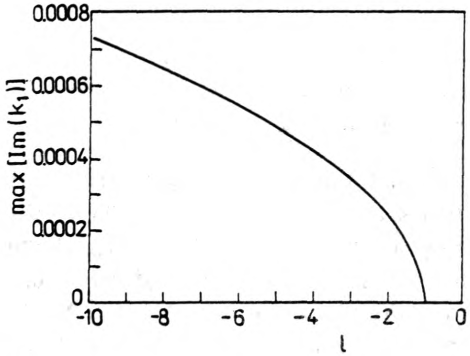
Fig. 7. As for Fig. 5, but $b = 0.99$ and $c = 0$. A case of larger amplitudes

Fig. 8. As for Fig. 7, but $l < l_{max}$

little energy to resist to destructing small amplitude perturbations. Resonances cause a transfer of the energy from the wave to the disturbances. This in effect brings the instability. Oppositely, large amplitude waves have so much energy to be robust with repect to the perturbations and we have got the stabilities.

Secondly, we take into consideration the case of the positive β, but now we get only periodic waves as solutions of the ENLSE. In Fig. 9, we see that although the instability rate is very small (10^{-4}) all periodic waves are unstable to this kind of perturbations. We have not observed the qualitative changes for larger amplitude waves.



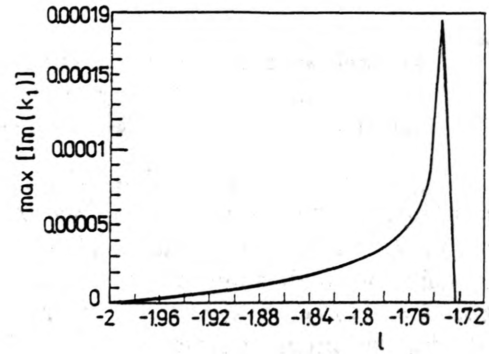


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Fig. 9. Maximum value of imaginary part of the dispersion relation solution for $\beta = 1$, $b = -3$ and $c = 1$. A case of periodic waves

Fig. 10. Maximum value of imaginary part of the dispersion relation solution for $\beta = -1$, $b = -1.5$ and $c = 2$. The "shock-wave" case. The lowest and greatest values of l correspond to a linear wave and the shock wave, respectively. The small amplitude limit

Finally, consider the case of negative nonlinear coefficient β, the "shock-wave" case. Here, we get a range of periodic waves bounded both by the linear wave and shock-wave like solutions of the ENLSE. In a small amplitude limit, we should recover the results obtained by INFELD and ROWLANDS [15], for the NLSE.

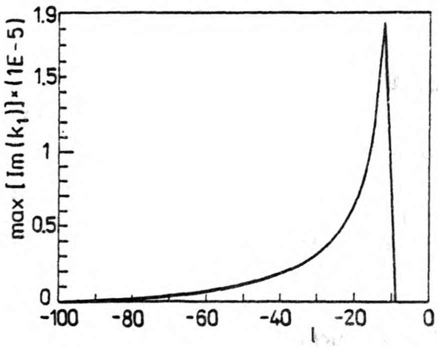

Fig. 11. As for Fig. 10, but $b = -1.99$, $c = 2$ and larger amplitudes waves

It has been found in this limit that all waves are weakly unstable but the rate of instability is about 10^{-4}. The shock wave has been found to be stable (Fig. 10). Practically, this means that waves are stable because we have calculated the integrals with an accuracy 10^{-4}. For larger amplitude waves, we have observed that waves are even more stable. The corresponding growth rate is about 10^{-5}. See Fig. 11.

7. Summary

Basing on the rigorous development of the nonlinear optics method, we have derived the nonlinear Schrödinger equation with the saturated exponential nonlinear term. Use has been made of the assumption of the exponential dependence of the refractive index $n(x,t)$ on the electric field $E(x,t)$. It has been shown that whenever the dispersive coefficient α is positive (anomalous dispersion) the waves are modulationally unstable when angular frequency of disturbances does not exceed the critical value. In this region of the angular frequency ω_0, it is impossible to carry on the experiment. The stability exists for $\alpha < 0$, however. And, thus, this is the best region of ω_0 to carry on the experiment.

The Infeld–Rowlands method has been developed to study stability of stationary waves, as solutions of the ENSE, with respect to small amplitude and long-period disturbances. The small amplitude soliton which is used as a carrier of an information in optical fibers has been found to be stable.

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References

[1] Mollenauer L. F., Stolen R. H., Gordon J. P., Phys. Rev. Lett. **45** (1980), 1095.
[2] Agrawal G. P., Baldeck P. L., Alfano R. R., Opt. Lett. **14** (1989), 137.
[3] Mollenauer L. F., Optics News, May (1986), 42.
[4] Hasegawa A., Kodama Y., Proc. IEEE **69** (1981), 1145.
[5] Kodama Y., J. Stat. Phys. **39** (1985), 597.
[6] Kodama Y., Hasegawa A., IEEE J. Quant. Electron. **QE-23** (1987), 510.
[7] Menyuk C. R., IEEE J. Quant. Electron. **QE-23** (1987), 174.
[8] Tzoar N., Jain M., Phys. Rev. **A 23** (1981), 1266.
[9] Anderson D., Lisak M., Opt. Lett. **9** (1984), 468.
[10] Shukla P. K., Rasmussen J. J., Opt. Lett. **11** (1986), 171.
[11] D'Evelyn M., Morales G. J., Phys. Fluids **21** (1978), 1997.
[12] Kaw P., Schmidt G., Wilcox T., Phys. Fluids **16** (1973), 1522.
[13] Sheerin J. P., Ong R. S. B., Phys. Lett. **63 A** (1977), 279.
[14] Karpman V. I., *Non-Linear Waves in Dispersive Media*, Pergamon Press, Oxford 1975, p. 26.
[15] Infeld E., Rowlands G., Z. Physik **B 27** (1980), 277.
[16] Piekara A. H., Proc. The VIth Conference *Quantum Electronic and Nonlinear Optics*, Poznań 1975, p. 9.
[17] McKinstrie C. J., Bingham R., Phys. Fluids **B 1** (1989), 230.
[18] Parkes E. J., J. Phys. A: Gen. **21** (1988), 2533.
[19] Tracy E. R., Chen H. H., Phys. Rev. **37 A** (1988), 815.

[20] INFELD E., ROWLANDS G., Proc. R. Soc. London A 366 (1979), 537.
[21] MURAWSKI K., STORER R., Wave Motion 41 (1989).

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