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**FURTHER EXAMPLES  
OF THE APPEARANCE OF MATRICES  
(AND THE ROLE THEY PLAY)  
IN THE COURSE OF THE ECONOMISTS' EDUCATION**

**Wojciech Rybicki**

**Abstract.** The paper makes up the third part of the series of articles aimed at establishing the usefulness of matrices for the study of contemporary economic sciences. The series was initiated by the present author in his previous articles of the subject (Rybicki, 2010, 2012). The themes discussed in the present article concern two areas: (i) the surface description of the Markov-type, continuous time stochastic dynamics (with special emphasis placed on “Poisson dynamics” and on families of transition probability matrices and intensity matrices – finite or denumerable spaces of states); (ii) the second is devoted to showing the role of eigenvectors in modeling the economic dynamics.

**Keywords:** Poissonian dynamics, queuing and risk models, intensity matrices, Kolmogorov equations, eigenvectors and eigenvalues of matrix of technology.

**1. Introduction**

This paper makes up the third part of the series of articles aimed at establishing the usefulness of matrices for the study of contemporary economic sciences. The present series was initiated by the author in his previous articles on the subject (Rybicki, 2010, 2012). The themes discussed in this article concern two areas: (i) the description of the stochastic dynamics of Markov-type systems (with the use of families of transition matrices, numbered by continuous time parameters, “running” along the positive part of  $t$ -axis), when special attention is placed on the Poissonian laws of motions, and intensity matrices – as generators of “huge powers” (like a mass, concentrated at some “aster-objects”), allowing them to create infinite chains of rules, governing the frames of randomness of trajectories (of classes of random process); (ii) the second one is devoted to exposing the role of

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**Wojciech Rybicki**

Department of Mathematics and Cybernetics, Wrocław University of Economics, Komandorska Street 118/120, 53-345 Wrocław, Poland.

E-mail: wojciech.rybicki@ue.wroc.pl

eigenvectors in modeling economic dynamics. In this context, there appear tasks of an important role in determining the so called stationary – with respect to the given transitions semi-group – distributions. On the other hand, they help to choose the optimal proportions of the sharing of components creating the input vectors in an individual firm’s management or “steering” the large economic systems (“optimal” means – guaranteeing the acceptable rates of return on the invested capital or development of the national economy as a whole). The considerations of the paper are completed by conclusions, containing also the announcement of the subject of a subsequent part of the series (entitled “The role of matrices in the contemporary education of students of economics – further remarks and examples of applications”), in which some general schemes (involving operations with the – sometimes generalized – matrices) are to be outlined. The author has prepared that article for print in the next issue of *Didactics of Mathematics* (Rybicki, 2013). The stochastic (Markov) integral kernels (as generators of Markov dynamics, as well as devices for the measurement and comparison of randomness of so called random projects – double stochastic operators) are also mentioned in the above paper, as well as some remarks of mixing (of parameterized families of probability distributions), the concept of monotone matrices (in the spirit of Schur ordering). The usefulness of matrices in the description of autoregressive schemes, the Youle-Wakker equations and some models of the financial market are noted there as well.

## 2. Matrices of the “Poissonian dynamics”

The classical model of a Markov chain (non-homogeneous case) with a countable state space makes up the following Poissonian “mechanics of changes” in discrete time. Actually, the discussed example appears as a (somewhat artificial) “extracting” discrete frames of time from the Poisson process on the positive half-line of reals (continuous time) – we will return to this process in a sequel. The transition probabilities are defined by the formulas:

$$P_m(i, j) = \begin{cases} \frac{\lambda_m^{j-i}}{(j-i)!} e^{-\lambda_m} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}, \quad (1)$$

for  $i, j = 1, 2, \dots$ ;  $m = 1, 2, \dots$ , where all the “lambdas” are positive (constant) parameters. If the requirement that  $\lambda_t \neq \lambda_s$ ; for  $t \neq s$  is fulfilled, the

non-homogeneity is guaranteed. When we put  $\lambda_i = \lambda$ , for all indexes (i), then the homogenous chain is obtained.

It may be of some interest to note the form of the matrix of transition probabilities in  $r$  ( $r > 1$ ) steps. The corresponding probabilities are easy to be obtained thanks to the fact that a distribution in each row of the matrix  $P_m$  is subject to the Poissonian rule

$$P_m = \begin{bmatrix} 0 & 0 & 0 & \dots & e^{-\lambda_m} & \lambda_m e^{-\lambda_m} & \frac{\lambda_m^2}{2!} e^{-\lambda_m} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & e^{-\lambda_m} & \lambda_m e^{-\lambda_m} & \frac{\lambda_m^2}{2!} e^{-\lambda_m} & \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{bmatrix}. \quad (2)$$

Hence  $P_{m,m+r} = [p_{m,m+r}(i, j)]$ ;  $m, r \in N$ ,  $i, j \in N$ , where

$$P_{m,m+r}(i, j) = \frac{(\lambda_m + \lambda_{m+1} + \dots + \lambda_{m+r-1})^{j-i}}{(j-i)!} e^{\lambda_m + \lambda_{m+1} + \dots + \lambda_{m+r-1}}.$$

So the matrix  $P_{m,m+r}$  is, of course, upper triangular as well.

Just this Markov chain was, at the beginning of the 20th century, applied by Swedish engineer Erlang to model the phone exchange (Erlang, 1917) and, practically at the same time, by another Swedish scientist, Lundberg, as a model of the stream of claims reported by an insurance company (Lundberg, 1903).

So these stimuli laid the foundations for the queuing theory and the theory of risk processes describing the dynamics of fluctuation of the side (or level) of insurance companies' capitals. Both of the above (sub)disciplines are present in the contemporary programs of a study of economics for their increasing significance in the business area. The origins of the subject concern the case when random variables  $Y_j$  ( $j \in N$ ) denote the number of calls incoming to the phone exchange from the moment  $t = 1$  to the moment  $t = j$ . Continuing to reflect the "ideal" randomness of "location" of subsequent calls on the positive (discrete) time axis there are assumed (or deduced from other primitive postulates): the independence of quantities of additional

signals observed during the interval  $j < t \leq j+1$  on the behavior of the process of calls before the instant  $j$  which are governed by Poisson distribution with parameter  $\lambda_j$  (the general, non-homogenous case).

The chain we are discussing is – sometimes – called the (stochastic) growth process: parallel to the passing of time the probability mass is shifting toward the states of higher numbers which may be interpreted as a (stochastic) tendency of increase of the global quantity of the considered systems (including – a very optimistic – version of the total economy level).

### 3. The Markov processes in continuous time and “their” families of transition probabilities

The above discussed chain makes a discrete time approximation to the variety of models describing the functioning of devices and systems in time, when the flows (or streams) of objects are flocking to be serviced. The mechanics of the arrival of these objects is assumed to be random. They can be of a multiple nature: elementary particles, observed in a molecular physics laboratory, calls registered by a phone exchange, items of products “crowding” to the next stage of the channel, claims reported by an insurance company or clients arriving at the ticket office, also the cars, lorries, boats, ships or planes. The somewhat idealized assumptions, such as a mutual independence of quantities reported in disjoint time intervals, identity of distributions of numbers of (say) calls coming at intervals of the same length and the exclusion of possibilities of the appearing portions of several (greater than one) objects at the same time (or in “infinitesimally short” intervals) leads to the classical Poisson processes – proceeding in continuous time  $X = (X_t; t \geq 0)$ , where

$$P(X_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda}; \quad k \in N, \quad \lambda > 0. \quad (3)$$

The above assumed properties of so-called stationarity (homogeneity) and independency of increments implies the Markovian property

$$P(X_{t+s} = l | X_t = k) = \begin{cases} \frac{(\lambda s)^{l-k}}{(l-k)!} e^{-\lambda s} & \text{for } l \geq k. \\ 0 & \end{cases} \quad (4)$$

So in this case we deal with a family of (stochastic) transitions matrices “numbered” by positive reals given by the formula:

$$P_s, s \geq 0, \quad (5)$$

where

$$P_t = [p_t(k, l); k, l \in N]; t \geq 0 \quad (6)$$

and

$$p_s(k, l) = P(X_{t+s} = l | X_t = k). \quad (7)$$

The process  $X$  is homogenous in time and space. The whole characteristic of this process is given by the family of transition matrices. It is worth noting the importance of the assumption excluding multiple signals which in turn prevent “explosions” – the discussed process is to be called ordinary Poisson process.

The parameter  $\lambda$  is called an intensity of process  $X$  and describes (roughly speaking) the “average probability of at least one signal appearing in the infinitesimal interval of time”. To be more precise, we can define (for the arbitrary stochastic stream with stationary increments) the intensity of such a stream as a limit

$$\lambda = \lim_{t \rightarrow 0} \frac{w(t)}{t}, \quad (8)$$

where  $w(t)$  denotes the probability of reporting at least one signal during the interval of length  $t$  (such limits always exist, although they may be improper ones (Kopociński, 1973)).

The related notions of great importance are the (general) matrices of intensities of transition for the state (say)  $k$ , to the state  $l$  (at the moment  $t$ ). We proceed with the introduction of them by some revivals of bases. The general theorems of the theory of Markov process (see e.g. Szekli, 1995) justify – under proper regularity conditions – identifying the arbitrary, continuous time, homogenous (in time) Markov process with finite or countable space of states with the family of transition probabilities. They are functions of the form  $P_h(i, j) = p_{t, t+h}(i, j) = P_{t, t+h}(i, \{j\})$  probabilities of “entering” the set  $\{j\}$  at the moment  $t + h$ , “departed” from the state  $i$  at the moment  $t$ .

The fact that the above family of generalized matrices plays a crucial role in a (complete) description of time homogenous Markov chains of this kind follows from the theorem below.

Theorem (Szekli 1995, p. 67, Corollary E)

It  $\{p_h(i, j); h \geq 0, i, j \in N\}$  is a family of measurable functions fulfilling

$$\sum_{i \in N} p_h(i, j) = 1, \text{ for all } h \geq 0, i \in N; \quad (\text{i})$$

$$p_h(i, j) = \sum_{k \in N} p_{t-h}(i, k) \cdot p_h(k, j), \text{ for all } 0 \leq h \leq t; i, j \in N \quad (\text{ii})$$

$$p_h(i, j) \geq 0; p_0(i, i) = 1; \text{ for all } i, j \in N, t \geq 0, \quad (\text{iii})$$

then there exists a Markov process  $(X_t; t \geq 0)$  such that, for all  $i, j \in N, h, t \geq 0$

$$P(X_{t+h} = j | X_t = i) = p_h(i, j),$$

and it has an arbitrary initial distribution  $\mu$ .

The above quoted theorem enables, in fact, to give the formal definition of Markov chains in continuous time (just like processes whose existence is ensured by the theorem). Let us change (slightly) only for a moment, the notation: by  $\{P_t; t \geq 0\}$  the (whole) family of transition probability matrices (of an infinite dimensions), will be denoted

$$P_t = [p_t(i, j)], \quad i, j \in N; t \geq 0 \quad (9)$$

symbol(s)  $p_t(i, j)$  should be interpreted as probabilities of transitions from the state  $i$  to the state  $j$  during the time interval of a length  $t$ . The property (of homogeneity) of processes allows us the “indexing” of the family with the use of merely a single parameter – the distance between the instants in mind. Suppose that for all entries of matrices  $P_t$  the following limits exist and are finite (Kopociński 1973, Szekli 1995)

$$q_{ij} = \lim_{t \rightarrow 0^+} \frac{p_{ij}(t)}{t}, \quad (10)$$

$$q_{ii} = \lim_{t \rightarrow 0^+} \frac{p_{ii}(t) - 1}{t}. \quad (11)$$

The matrix  $Q = [q_{ij}] i, j \in N$  is called the intensity matrix of this family (or the intensity matrix of the corresponding Markov chain).

A glance at the formulas defining intensities, suggests the possibility of associating them with some kind of (natural) derivatives (at zero) of transition probabilities. This circumstance is of much more importance than a solely “graphical” similarity. For “small”  $t$  we may write

$$P_t(i, j) = q_{ij}t + o(t) \sim q_{ij} \cdot t, \text{ and} \quad (12)$$

$$p_t(i, i) = 1 + q_{ii}t + o(t) \sim 1 + q_{ii} \cdot t. \quad (13)$$

The above relations express the property of “infinitesimal proportionality” of transitions probabilities to (corresponding) intensities.

The above observations can be summed up in the following notation

$$Q = \left. \frac{dP_t}{dt} \right|_{t=0}. \quad (14)$$

On this occasion we recall some terminology. The intensity matrix  $Q$  is called regular if  $\sum_{j \neq i} q_{ij} = -q_{ii}$ . If in addition,  $\sup(-q_{ii}) < \infty$ , then the adjective “uniform” is added. It is worth noting the “finite case” – when an intensity matrix is easily “visualized” as a picture

$$Q = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0n} \\ q_{10} & q_{11} & \cdots & q_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n0} & q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

these properties (regularity and uniformity) are always fulfilled. The *clou* of interest and the essence of the role of intensity matrices, is “potentially hidden in them”: the single matrix suffices to reveal and determine the whole stochastic mechanics of the chain in mind (Kopociński, 1973; Szekli, 1995).

A powerful device is provided by the famous Kolmogorov equations together with expanding in the power series of exponential (general, matrix-like) function (the convergence of matrices appearing in the formulas is well defined, by adopting “natural” norms).

Theorem (Theorem *F*, Szekli 1995, p. 68; Feller 1969; Kopociński, 1973).

If  $Q$  is a uniform intensity matrix, then the formula

$$P_t = e^{Qt} = \sum_{n=0}^{\infty} \frac{Q^n t^n}{n!} \quad (15)$$

defines a Markov transition functions family, which is the unique solution of the differential equations:

$$\frac{dP}{dt} = Q \cdot P; \quad t \geq 0 \quad (16)$$

and

$$\frac{dP_t}{dt} = P_t \cdot Q; \quad t \geq 0 \quad (17)$$

with the initial condition  $P_0 = I$  ( $I$  denotes the identity matrix and the above differential equations are called backward and forward Kolmogorov equations, respectively).

The possibility of a “reconstruction” of the (uncountable, continuously indexed) family of probability rules, governing the random movement on the non-negative half-line of the time, from the generator  $Q$ , is a very strong property – in the spirit of (analytical) extending of analytical functions from small neighborhood of “initial” point to the whole domain (reals or complex numbers).

Moving on to the end of the present point, we can stress the role of the semi-group character of the family of transition operators (represented in the discussed case by matrices) and the continuity (with respect to the time parameter) of this family. One word may be added also on the generalizations of the discussed subject: the classical topics (coming back to Feller’s early work (Feller, 1941) and the seminal Hille-Yosida type theorems, concern the notion of the so called infinitesimal generator of continuous semi groups – of linear operators, Szekli, 1995, p. 69). When applying to the (very) general state space the Markov process, such generators – according to their “label” – generate the whole (continuous time) process: the totality of all transition probabilities (functions, their densities etc.). But those theoretical considerations go beyond the scope of this paper – devoted to the role of matrices in the “quantitative education” of students of economics.

In the next point we return to the “ordinary” (finite dimensional) matrices and outline the certain reasoning, enlightening the role of eigenvectors (and eigenvalues) of matrices in the modeling of (simple) economic growth processes.

#### 4. Why the eigenvectors and eigenvalues are needed

In the present point two applications of matrices to model the economic dynamics will be shown. We have got “accustomed” to the role matrices play as representations of rules of motions. Now we are going to “extract”

the specific aspects of “steering” of the economic process – the special role played here by eigenvectors and eigenvalues of (square, finite size) matrices.

Let us recall some basic definitions, following the English edition of the monograph of Kurosh (see f.e. (Kurosh, 1972, pp. 199-203). Let the (linear) operation  $\mathbf{a} : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be represented by the square matrix  $\mathbf{A}$  of size  $n$ . It simply means that  $\mathbf{a}(\bar{x}) = \mathbf{A}\bar{x}$   $x \in \mathbf{R}^n$ , where vectors on the right are assumed to be column vectors. If a non-zero vector  $b \in \mathbf{R}^n$  is carried by its transformation into an image-vector of the form

$$\mathbf{a}(b) = \lambda \cdot b, \quad (18)$$

where  $\lambda$  is some real number, then the vector  $b$  is called an eigenvector of the transformation  $\mathbf{a}$  and, at the same time, an eigenvector of the matrix  $\mathbf{A}$ . At the matrix notation we obtain

$$\mathbf{A} \cdot b = \lambda b. \quad (19)$$

We also say that  $\lambda$  is an eigenvalue of the transformation (as well as of the matrix  $\mathbf{A}$ ) corresponding to the vector  $b$ . The eigenvectors and “their” eigenvalues are (relatively) simply obtained, according to the classical facts (and methods) concerning solutions (existence and uniqueness) of systems of linear equations (more basically, they follow from elementary properties of linear operations in finite-dimensional vector spaces). Rewriting the last equations leads to the chain of its subsequent shapes

$$\mathbf{A}b - \lambda b = \mathbf{A}b - \lambda \mathbf{I}b = (\mathbf{A}b - \lambda \mathbf{I})b = 0. \quad (20)$$

The above equation has a non-zero (vector) solution if determinant  $\det(\mathbf{A} - \lambda \mathbf{I})$  equals zero. So the condition

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (21)$$

serves determining (a set of) “lambdas” – characteristic roots of matrix  $\mathbf{A}$ , (in other words – roots of a characteristic polynomial – in  $\lambda$  – of this matrix). Knowing the (concrete) eigenvalue  $\lambda$ , one may find the set of eigenvectors associated with  $\lambda$ . It is a rule that they form a subspace of  $\mathbf{R}^n$  without “deleted” (“abandoned”) vector zero. After presenting the ideas we pass on to the first example.

Consider the (extremely simplified) model of a business cycle (or economic cycle), in a “stochastic costume” – we follow the example discussed in the textbook (Ostojka-Ostaszewski, 2006, pp. 234-235). They are assumed to be taken as merely two possible states of the economy: a “revival” (it is

the “sum” of classical periods – recovery and expansion) and a “recession” (summarizing the “classic” phases of a full cycle: crisis and recession). The “random moves” of the economy from state to state are governed by the transition matrix (of the homogenous two-state, discrete time, Markov chain)

$$P = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}, \quad (22)$$

where the state (say) “revival” is associated with the first row and first column, while the second row and second column correspond to “recession”. Let us assume  $0 < p < 1$  and  $0 < q < 1$ . So the economy goes to “the revival” at the next moment starting from “the revival” at present, with probability  $p$ , gets into “the recession” with complementary probability  $1 - p$ ; the transition from “recession” to the “revival” occurs with probability  $1 - q$  and remaining in “recession” – with probability  $q$ .

It seems to be proper to recall once more the notions connected with the studied subject. One of the fundamental questions in the theory of the Markov process is a problem of the existence of the limit behavior of instantaneous (marginal) distributions. Under natural, mild conditions put on the entries of transition probabilities matrix, there exist limits

$$\lim_{k \rightarrow \infty} p_k(i, j) = \pi_j^* \quad \text{for all } i, j \in N \quad (23)$$

and they are independent of initial distributions [Szekli, 1995; Kopociński, 1973]. The property may be classified as a statement from the field of the ergodic theory. The second notion concerns properties belonging to the area of stationarity (of the stochastic processes in a strict sense). As is known, the process is termed strictly stationary if its finite dimensional distributions (of the same dimension) are invariant with respect to translations of sets of indexes along the time axis. The related notion is the so called invariancy (probability) measure  $\pi$  on the state space of process, with respect to the family of transition probabilities matrices  $\mathcal{P} = \{P_t, t \in \mathbf{T}\}$  (also called the stationarity of this distribution with respect to  $\mathcal{P}$ ) and is expressed in the formal requirement.

$$\pi \cdot P_t = \pi \quad \text{for all } t \in \mathbf{T}. \quad (24)$$

Remember, by the way, that (in the case of discrete time Markov chains) their defining relation takes the form

$$\pi_t \cdot P = \pi_{t+1} \quad (25)$$

( $t \in N$ , both  $\pi_t$  and  $\pi_{t+1}$  are row probability vectors).

Denoting by  $T$  the “time” allows us to treat simultaneously the case of discrete and continuous time. The discrete time version of the requirement (24) is

$$\pi P = \pi. \quad (26)$$

The remaining statements we want to recall are: the limit vector  $\pi^* = (\pi_j^*)$  is stationary with respect to the  $P$  and, more generally, the stationarity with respect to the  $P$  is equivalent to stationarity of the chain (strict sense) if the limiting distribution coincides with the initial distribution  $\pi^0$  (at the moment  $t = 0$ ). In such a case  $\pi^0 = \pi = \pi^*$ .

The relation (26) says, in fact, that  $\pi$  is the eigenvector of matrix  $P$  corresponding to the eigenvalue 1. Coming back to the simple model of business cycle, we obtain the characteristic equation (of matrix  $P$  appearing in the case):

$$\begin{bmatrix} p - \lambda & 1 - q \\ 1 - p & q - \lambda \end{bmatrix} = 0. \quad (27)$$

After calculations the equation

$$(\lambda - 1) [\lambda + (p + q - 1)] = 0 \quad (28)$$

is obtained. So  $\lambda = 1$  is in fact the eigenvalue of  $P$ . The secondary school algebra leads to the general form of corresponding family of eigenvectors

$$[x, y] = \alpha[1 - q, 1 - p]. \quad (29)$$

After normalizing (by summing components to units) we obtain

$$\pi = [\pi_1, \pi_2] \quad , \quad (30)$$

where

$$\pi_1 = \left( 1 + \frac{1-p}{1-q} \right)^{-1}; \quad \pi_2 = 1 - \pi_1.$$

So  $\pi_1 > 0.5$  if  $p > q$ , that in turn may be interpreted as a condition for the periods of revival prevail – in the long term (which sounds very acceptable from the intuitive perspective).

The last (but – anyway – the least!) example is taken from the textbook on linear algebra (Smoluk, 2007). Let  $P$  denote the matrix of technology in a linear (discrete time or – even – “quasi-static”) model of economic growth (or, respectively, “time-less” production). We will follow the brilliant arguments of Professor Smoluk, showing the significance of the choice of proportions of factors creating the input of process.

Let us begin with the general scheme of the linear transformations of the economic system from the initial vector of states into vectors in the “next” moment. The justification of assuming the linearity of the considered (modeled) transformation is a “rationale one”: the outputs “should be” (approximately) proportional to the inputs – this makes an innocuous simplification of the “real economic world” – but within corollaries concerning the functioning of complex economic systems, have been discovered (“as a reward”) the number of interesting (sometimes – fundamental, as Leontief-type models: Leontief, 1941; Gale, 1969; Malawski et al., 1997). The multiple super-positions of these operations generate the path of economic growth (in the macro, multi-sector scale) or the sequence of subsequent outputs of a production system, say, a firm as a whole or an individual business or even an “isolated” enterprise accomplished with the use of some feasible technology) resulted as the consequences of the coming inputs (microeconomic context). It should be noted that the output of the  $n$ -th stage of a process becomes the input of the stage of a number  $n + 1$ .

Let us assume, for simplicity, that the technologies (or algorithms of distribution of resources among branches of economy – input flows) remain invariant (as discrete time goes). So the single input – output matrix and the initial decision on the division of means at disposal determine (theoretically) the “turnpike” of a development.

The decision makers (central planners or managers, controlling the production processes) are interested (first of all) in maximizing the values of some development coefficients (say, rate of growth) and finding optimal paths of growth, given a matrix  $\mathbf{A}$  – of technological characteristics (or input – output) matrix.

They aim to attain the optimal relations between the costs and benefits in the discussed case: between input(s) and output(s) (vectors). For a given matrix (Smoluk, 2007)

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{bmatrix} \quad (31)$$

the characteristic polynomial takes a simple form

$$\chi(\lambda) = |A - \lambda I| = \begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{3} & 1 - \lambda \end{vmatrix}, \quad (32)$$

hence the characteristic equation may be written as a

$$(1 - \lambda)^2 = \frac{1}{6}, \quad (33)$$

from which, directly, follows that eigenvalues of  $\mathbf{A}$  are

$$\lambda_1 = 1 - \frac{1}{\sqrt{6}}, \quad \lambda_2 = 1 + \frac{1}{\sqrt{6}}. \quad (34)$$

To calculate the eigenvectors corresponding to these roots ( $\lambda_1$  and  $\lambda_2$ ), one needs to solve two systems of homogeneous linear equations (respectively)

$$\begin{bmatrix} 1 - \left(1 - \frac{1}{\sqrt{6}}\right) & \frac{1}{2} \\ \frac{1}{3} & 1 - \left(1 - \frac{1}{\sqrt{6}}\right) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (35)$$

and

$$\begin{bmatrix} 1 - \left(1 + \frac{1}{\sqrt{6}}\right) & \frac{1}{2} \\ \frac{1}{3} & 1 - \left(1 + \frac{1}{\sqrt{6}}\right) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (36)$$

The solutions of these (undetermined) equations have the form (of the univariate subspaces, without zeros):

$$v_1 = \{t(\sqrt{3}, -\sqrt{2}); t \in R\} \quad (37)$$

and

$$v_2 = \{t(3, \sqrt{6}); t \in R\}, \quad \text{respectively.} \quad (38)$$

Inputs in the economy, described by matrix  $\mathbf{A}$ , are proportional to the eigenvectors:  $v \in R^2$  results in “going out”  $\mathbf{A}v = \lambda v$ .

It is clear that being in front of the choice between different (potential) input eigenvalues,  $v_1$  or  $v_2$ , the rational decision maker will choose the vector  $v_2$ , for just this vector corresponds to the greatest eigenvalue of the matrix  $\mathbf{A}$  (the number  $\lambda_2$ , additionally it is greater than one) which is a property of extraordinary importance in the discussed problem.

Iterating this strategy we will obtain for  $n$  years output equal to  $\lambda_2^n v_2$ , so the economy flourishes and the welfare of society approaches infinity. On the other hand, the choice of  $v_1$  to be the vector of the initial input leads to the recession and even the ruin of the economy. The reason is equally clear as that of the previously discussed situation. When  $\lambda_1 < 1$ , then after  $n$  years the economy unavoidably becomes worse and worse:  $\lambda_1^n$  tends to zero when  $n$  approaches infinity. The author, as well as the reader, is aware of the idealization (touching the limits of a joke) from which the model suffers. Anyway, the choice of the (proper) structure of economy plays a decisive role in its development.

## 5. Conclusions

The matrices appear very often in the problems of (in a wide sense) economic character. They, as well as some of their generalizations, work in describing schedules of movement of Markov processes in continuous time, they make up the role of representations and “hearts” of stochastic (Markovian) semi-group of operators – the special role played by the intensity matrices of finite or countable dimensions: they generate the whole (stochastic) behavior of the processes, on the other hand they are – in a sense – “manageable”, thanks to the Kolmogorov equations, and have a convincing intuitive meaning. It should be remembered that for the students it is compulsory to become familiarized with the Markov processes when they learn elements of operation research (queuing, renewal and reliability theories) and encounter problems of insurance risk processes or some stochastic finance (and elements of financial engineering).

The role of eigenvalues of matrices – in the context of describing economic dynamics – should also be pointed out. They decide – at least in the theoretical models – on the efficiency of production (multi-sector inputs) and/or: the directions and speed of development of the great aggregates of the national economy as a whole. One common corollary may be extracted

from the considerations (which does not aspire to the range of a “discovery”, but is merely aimed as a methodological (didactic) reflection and a kind of “hint’ (to be taken into account): matrices (like numbers, but – may be – in a significantly greater generality) make up the “laboratory example” of functioning of abstraction processes – inevitable for any mental activity, simplifying things and extracting their essence. On the other hand, they are – simply – “easy, friendly and convincing” for beginners, and – as such – can be successfully exploited during the process of teaching future managers, directors and (perhaps) politicians forced to possess some knowledge of the true mechanics of economics. Experience suggests that some subtle idea might be “hidden” and “smuggled”, when (cleverly) “dressed in costumes of naive tables”.

Moving on to the end of the concluding remarks (and the article), the author would like to announce some items, which will be presented in the next paper of the series (mentioned in the introduction, Rybicki, 2013). Some generalizations of the notion of “classical” matrix are to be shown: stochastic (Markov) integral kernels (as generators of Markov dynamics, as well as devices for measurement and comparisons of randomness of the so called random projects – double stochastic operators are mentioned), some remarks of mixing (of parameterized families of probability distributions), the concept of monotone matrices (in the spirit of Schur ordering). The usefulness of matrices in the description of autoregressive schemes, the Youle-Wakker equations and some models of the financial market are noted there as well. Some information on covariance and correlation (including auto-correlation) matrices and sequences are also given – referring to the Gramm matrices and general positive definite functions.

The themes discussed in the present article concern two areas: (i) the description the stochastic dynamics of Markov-type systems (with the use of families of transition matrices, numbered by continuous time parameters, “running” along the positive part of  $t$ -axis, when special attention is placed on the Poissonian laws of motions, and intensity matrices - allowing them to create the infinite chains of rules, governing random process; (ii) the second one is devoted to exposing the role of eigenvectors in modeling the economic dynamics. In this context there appear tasks of an important role in determining the so called stationary – with respect to the given transitions semi-group – distributions. On the other hand, they help to choose the optimal proportions of the sharing of components creating the input vectors in the theory of firms, as well as the theory of economic growth.

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