DIDACTICS
OF
MATHEMATICS

7(11)

The Publishing House
of the Wrocław University of Economics
Wrocław 2010

Editors
Janusz Łyko
Antoni Smoluk

Referee
Marian Matłoka
(Uniwersytet Ekonomiczny w Poznaniu)

Proof reading
Agnieszka Flasińska

Setting
Elżbieta Szlachcic

Cover design
Robert Mazurczyk

Front cover painting: W. Tank, *Sower*
(private collection)

PL ISSN 1733-7941

Print run: 200 copies

TABLE OF CONTENTS

D I D A C T I C S   O F   M A T H E M A T I C S

No. 7 (11) 2010

APPLICATIONS OF THE INTEGRAL IN ECONOMICS
A FEW SIMPLE EXAMPLES FOR FIRST-YEAR STUDENTS

Marek Biernacki

Abstract. One fundamental problem encountered when effectively teaching mathematics to students of economics is communication with learners that lack an adequate background. In the past, more than 66 per cent of freshmen enrolled at one of the faculties of our university did not take their school-leaving examinations in mathematics, even at the elementary level. One method to encourage students to better learning includes the examples of areas in economics where they can apply mathematical concepts. Usually, the course in mathematics is relatively short and offered in the first semester, therefore it is not quite easy to actually introduce necessary economic concepts, even if only intuitively. Nevertheless it is worth trying. The paper gives several simple and useful examples of an integral's applications in economics, such as estimating resources based on the intensity function of a flow; calculating a consumer's surplus; the time value of capital; comparing levels of social wealth.

Keywords: consumer surplus, integral, producer surplus.

1. Introduction

The Faculty of Economic Sciences at the Wrocław University of Economics reduced a number of hours in mathematics for students of management from 120 hours (60 hours in two semesters) to 45 hours (30 hours of lectures and 15 hours of classes). The reduction was defended by the expectation that less obligatory hours would result in more individual work of students. The real life however refuted the premise, at least as regards the first-year students of our university. The observations and interviews of our students demonstrate that they do not even meet a requirement to do their own work before obligatory classes, not to mention supplementing, deepening or widening material in mathematics.

Marek Biernacki
Department of Mathematics, Wrocław University of Economics, Komandorska Street 118/120, 53-345 Wrocław, Poland.
e-mail: marek.biernacki@ue.wroc.pl

A survey was conducted among first-year students at the end of a one-semester course in mathematics in 2008. The respondents were asked six questions concerning the lecture based on a scale from 1 to 5. “Correct” graphs should be left-skewed: a good assessment of lecture, and low variation: a good lecture for most of students. Figure 1 and Table 1 show the responses to the three questions:

1. Suitability of literature recommended for the course.
2. Punctuality of instructors.
3. Consistency of syllabus with the major field of study.

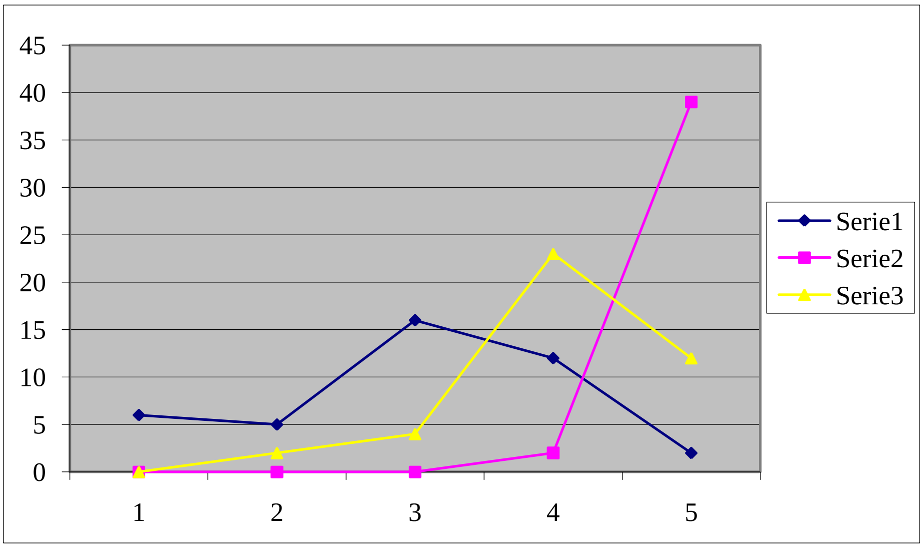

Fig. 1. An assessment of the lecture based on the survey

Source: author’s own calculations.

Table 1. Averages, standard deviations and variation of responses

Item	1	2	3
Average	2.97561	4.95122	4.097561
Standard deviation	1.106522	0.218085	0.768273
Coefficient of variation	0.371864	0.044047	0.187495

Source: author’s own calculations.

The literature recommended to the students at the beginning of the course included among others the following handbooks: Ostoja-Ostaszewski, *Mathematics in economics* (vol. 1 and 2); Bednarski, *Elements of mathematics in economic sciences*; Antoniewicz, Misztal, *Mathematics for students of* *economics*; Smoluk, *Basic mathematical analysis*. The practical part of the course was based on collections of problems by Krysicki and Włodarski in analysis, by Gewert and Skoczylas, and by Jurlewicz and Skoczylas in linear algebra.

Unfortunately, students' responses in the survey and conversations during classes and consulting hours showed that they did not read the textbooks, therefore they found it difficult to supplement their knowledge, or to deepen and fully understand the lecture so as to be well prepared for class sessions aimed at better understanding of the material. Hence, it was key to develop such methods that would positively encourage students to do own work. The simplest way is to show them some easy and attractive applications into what is of primary interest to them, i.e., in economics, especially as the Ministry expects "applications of mathematics in management" among learning outcomes, skills and competencies in this field of study. Because the lecture in mathematics is offered to students in the first term, we face some difficulties using necessary economic concepts (at least intuitively), nevertheless, my experience proved that it is worthwhile trying. While there are a number of interesting applications of differential calculus (elasticity, optimization, substitution rates, etc.), integral calculus is not that easy to be illustrated to first-year students. It is worth noting that a very good textbook in mathematics for students of economics by Adam Ostoja-Ostaszewski does not cover integral calculus at all.

Therefore, this short paper offers some introductory examples of how to apply integral calculus in diverse areas of economics, with problems that are accessible by a first-year student such as: estimation of resources using flow intensity function; index of income concentration, consumer's surplus calculation; investment value over time; value of economic welfare and comparing social welfares.

2. Examples of applications of integral calculus in economics

2.1. Mean value of a function on the interval [*a*, *b*]:

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem 1. Let us assume that the value of a US dollar expressed in the Polish złoty from day 1 to day 30 of a certain month is represented by a function of time:

$$f(t) = 2{,}5 + 0{,}5 \cdot \sin\left(\tfrac{\pi}{15} t\right).$$

Find the mean value of a US dollar in this month.

2.2. A definite integral as the area under a curve

The operations of a certain firm are modelled by the function $k(t)$ that is continuous with respect to time and represents the cost of material used in the production process. At a given moment t, the value $k(t)dt$ can be interpreted as the cost incurred in a very short period (t, $t + dt$). Hence, a total value of cost on the interval [0, T] will be approximately equal to:

$$\int_0^T k(t)dt$$

in agreement with a geometric interpretation of the Riemann integral, i.e., the area of the region bounded by the graph of the function $k(t)$ on the interval [0, T].

Let us now assume that the period is long enough so that we should consider the time value of money. In that case, a concept of compound interest should be earlier explained, together with continuous compounding based on the definition of Euler's constant, and also a general concept of the time value of capital (or money).

Using the model of the time value of capital, given the compound interest rate r, we calculate a total value of capital at time T and at time 0 equal to

$$K(T) = \int_0^T k(t)(1+r)^{T-t}\, dt$$

and

$$K(0) = \int_0^T k(t)(1+r)^{-t}\, dt.$$

Using a model of the time value of capital, given continuous compounding at the rate r, we obtain a total value of capital at time T and at time 0 equal to:

$$K(T) = \int_0^T k(t) e^{r(T-t)} dt$$

and

$$K(0) = \int_0^T k(t) e^{-rt}\, dt\,.$$

2.3. Income concentration index – the Gini coefficient *G*

$$G = 2\int_0^1 (x - L(x))dx$$

where $L(x)$ is the Lorenz curve defined as follows. Let us assume that a vector of incomes $x = (x_1, ..., x_n)$ is arranged in non-decreasing order: $x_1 \le x_2 \le ... \le x_n$. The empirical Lorenz function is generated by points, whose first coordinates are numbers i/n, where $i = 0, 1, \dots, n$; n is a fixed number, and second coordinates are determined as follows: $L(0) = 0$ and

$$L\left(\frac{i}{n}\right) = \frac{s_i}{s_n},$$

where $s_i = x_1 + x_2 + ... + x_i$. The Lorenz curve is defined at all points $p \in (0, 1)$ through linear interpolation. One can show that $L'(x) > 0$ and $L''(x) > 0$, $L(0) = 0$ and $L(1) = 1$.

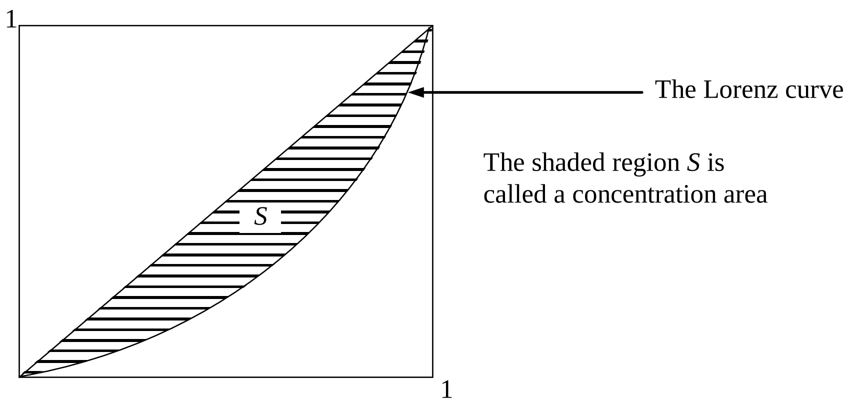

Fig. 2. The Gini coefficient equals two concentration areas (2*S*). Thus,

Source: author's own calculations.

$$G = 2\int_0^1 (x - L(x))dx.$$

When the value of the Gini coefficient is close to 0, the underlying distribution is almost uniform, whereas the value close to 1 indicates a maximal inequality, i.e., a total wealth of a population is concentrated in hands of one man.

Problem 2. The Lorenz curve of income distribution within a certain group is given by the formula:

$$G = 2\int_0^1 (x - L(x))dx.$$

Determine the degree of equality of income distribution.

Solution:

$$G = 2\int_0^1 (x - (0,8x^2 + 0,2x))dx = 2\int_0^1 (-0,8x^2 + 0,8x)dx = 0,27.$$

Because G is relatively small, we conclude that the given income distribution is fairly uniform.

Problem 3. The Lorenz functions L_1 and L_2 of income in a population of teachers and administration employees of a certain school are given by formulas:

$$L_1(x) = 0,6x^2 + 0,4x, \qquad L_2(x) = 0,3x^2 + 0,7x.$$

Which group of employees is more uniformly distributed?

$$G_1 = 2\int_0^1 (x - (0,6x^2 + 0,4x))dx = 2\int_0^1 (-0,6x^2 + 0,6x)dx = 0,2,$$

$$G_2 = 2\int_0^1 (x - (0,3x^2 + 0,7x)dx = 2\int_0^1 (-0,3x^2 + 0,3x)dx = 0,1.$$

We conclude that income is more uniformly distributed in the group of school's administration workers.

2.4. The surplus of consumer and producer

The figure illustrates the relationship between a product's price and its supply. If a point s refers to the equilibrium price $c(s) = p^*$, then the value of products sold at the equilibrium price equals $(s - 1)\cdot c(s)$. On the other hand though, when the product is sold differently, it presents a higher value to a consumer, i.e., when the first piece of the product is

$$\int_1^s c(u)du$$

sold at $c(1)$, the second one at $c(2)$, and so on. Then the total value of sales will amount to $c(1) + c(2) + \ldots + c(s)$ and, according to the geometric interpretation of the Riemann integral it will be close to the value of:

$$\text{the area } \int_1^s c(u)du - (s-1)\cdot c(s)$$

denoted by *nk* in Fig. 3 is called a consumer's surplus; *wt* denotes the value of the product.

The economists are interested in the changes of consumer's surplus resulting from the price changes, say, from price c_1 to c_2. To calculate such a change, we should take the demand function showing how demand depends on a price: $u(c) = c^{-1}(u)$ and the change of a consumer's surplus is equal

$$\int_{c_1}^{c_2} u(c)dc.$$

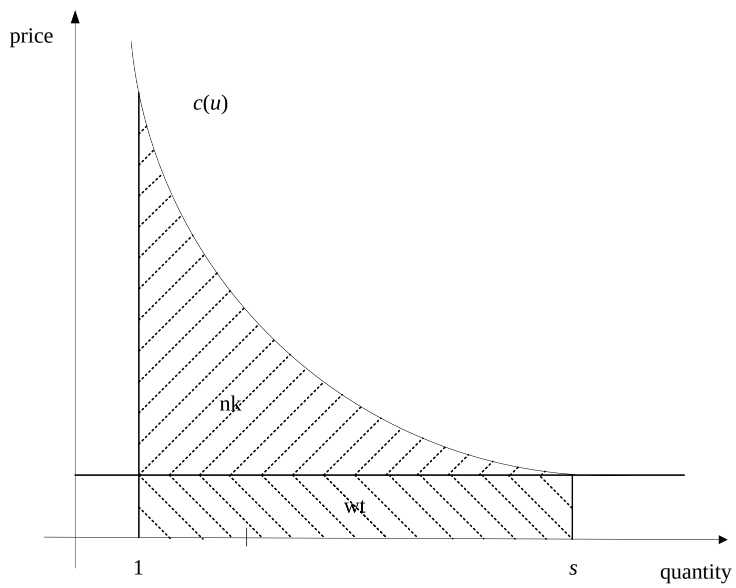

Fig. 3. A consumer's surplus and the value of product

Source: author's own calculation.

Problem 4. Let us assume that the demand function of a commodity with respect to its price is given by $u(c) = 2/c$ and that its price increased from PLN5 to PLN6. What is the change in a consumer's surplus (or how much a consumer will lose due to that increase)?

The change in the surplus (consumer's loss) equals:

$$\int_5^6 \frac{2}{x}dx = 2\ln(x)\Big|_5^6 = 2\big(\ln(6)-\ln(5)\big) = 2\ln\left(\frac{6}{5}\right).$$

Figure 4 depicts a supply curve or amounts of a commodity supplied at each price. Similarly to the demand curve, the area above the supply curve up to $p = p^*$ is called a producer's surplus and denoted by *np*. That surplus is achieved by a producer when he is able to sell x^* units of his commodity at price p^*; hence:

$$np = \int_0^{x^*} \big(p^* - S(x)\big)dx.$$

Problem 5. Given the function $p = s(q) = 0.2q$ of a commodity's price p depending on its supply and the function $p = c(q) = 5/q$ of a commodity's price depending on its demand, find a consumer's surplus and a producer's surplus.

Solution:

$$0,2q = \frac{5}{q} \Leftrightarrow q = 5.$$

If $q = 5$, then $p^* = 1$.

Thus,

$$nk = \int_1^5 \frac{5}{x}dx - (5-1)\cdot 1 = 5\ln(x)\Big|_1^5 - 4 = 5\ln(5) - 4;$$

$$np = \int_0^5 \left(1 - \frac{x}{5}\right)dx = \left(x - \frac{x^2}{10}\right)\Big|_0^5 = 2,5.$$

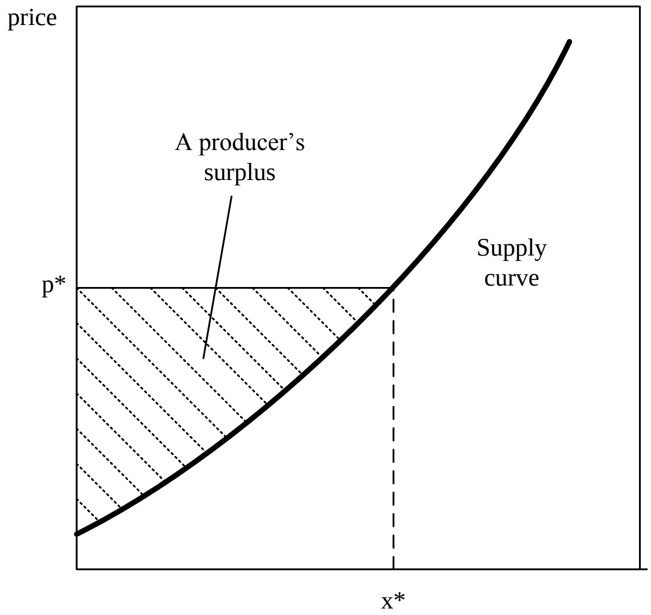

Fig. 4. A producer's surplus

Source: author's own calculation.

2.5. The probability density function of a continuous random variable *X*

A function $f(x)$ is called the probability density function of a random variable X, when it satisfies the following conditions:

$$\forall x \in R \quad f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Then the probability that the value of a random variable X will belong to the interval $[a, b]$ equals:

$$P(a < X < b) = \int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is a primitive function of a random variable X and – at the same time – its cumulative distribution function. In addition, we have:

$$\int_{-\infty}^{\infty} x \cdot f(x)\,dx = \mu,$$

where μ is the expected value of a random variable X. If $f(x)$ is a density function of income distribution in a certain population and $u(x)$ is a utility function of demand, then $u(x)$ satisfies the following conditions: $u'(x) > 0$ and $u''(x) < 0$. Subsequently,

$$\int_0^\infty u(x)\, f(x)\, dx = SW$$

will be called an economic welfare of a given population. In 1973, A.K. Sen introduced the following measure of welfare that is sensitive against inequality of income distribution: $IS_F = \mu_F(1 - G_F)$, where μ_F and G_F are the mean value and the Gini coefficient, respectively, of income distribution with the cumulative distribution function F.

As an exercise for more inquisitive students, we can prove that a following implication holds:

$$IS_F \le IS_G \Rightarrow SW_F \le SW_G.$$

Literature

Bednarski T. (2004). *Elementy matematyki w naukach ekonomicznych* [Elements of mathematics in economic sciences]. Oficyna Ekonomiczna. Kraków.

Biernacki M. (2006). *Porządki generowane krzywą Lorenza* [Orders generated by the Lorenz curve]. Mathematical Economics 3(10). Wydawnictwo Uniwersytetu Ekonomicznego we Wrocławiu. Wrocław. Pp. 125-134.

Biernacki M. (2009). *Effectiveness of mathematical education*. Didactics of Mathematics 5-6 (9-10). Wydawnictwo Uniwersytetu Ekonomicznego we Wrocławiu. Wrocław. Pp. 19-32.

Varian H. (1997). *Mikroekonomia* [Microeconomics]. PWN. Warszawa.