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PROCESS DENSITY FUNCTIONS IN THE PROBLEM OF THE IDENTIFICATION OF A BARRIER IN THE FUNCTIONING OF A CERTAIN INVENTORY STORAGE AND ISSUE SYSTEM

The subject matter of the investigation is a barrier to the functioning of a certain inventory system in the case where the storage input is a non-aggregated dynamic-parameter process. The authors derive a system of differential equations satisfied by probability distribution density functions for the intermediate states of subsystem L . The system of equations expresses relations between the densities and the parameters of the functioning of the transport subsystem and the parameters of the products supply process.

Key words: *inventory system, barrier, transport, system of differential equations*

1. Introduction

The quantitative analysis of barriers to the functioning of actual inventory systems is of essential importance in the process of the rationalization of such systems (cf., e.g., [1], [2], [4]–[6]).

The subject matter of our investigation is a barrier existing in the following inventory storage and issue system.

A receiver E (e.g., a power plant), whose operation depends on the constant supply of a units of a product (e.g., coal), is supplied continuously (e.g., by conveyor belt, pipeline, power supply line) with a production flow $y(t)$, generated by a production subsystem \tilde{P} . Random changes of the process $y(t)$ and unplanned interruptions (fail-

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ures) of the transport subsystem \tilde{T} are factors that result in a decrease of the efficiency of the system. The efficiency may be increased – and at the same time the probability of interruptions in the supply of the product to the receiver E may be reduced – by placing a storage container M with a certain volume V near the receiver E . The product flow $y(t)$ is stored in subsystem M if the storage level is less than V and $y(t) > a$. If the instantaneous contents of M are equal to V and $y(t) > a$, the flow $y(t)$ is reduced to the level of a . If M is empty and $y(t) < a$, the situation is disadvantageous to the receiver E ; the determination of the probability of such an occurrence is of essential practical importance.

Subsystem \tilde{P} may be exemplified by a group of excavators. The input $w(t)$ to subsystem M , which describes the process of the supply of the product to M , will be considered in a nonaggregated variant: $w(t) = y(t)v(t)$; the process $w(t)$ explicitly reflects both subsystem \tilde{P} (the product flow $y(t)$) and subsystem \tilde{T} (the process $v(t)$ describes the functioning of the transport subsystem).

Let us consider a variant where the production $y(t)$ is supplied from subsystem \tilde{P} to subsystem M in a continuous way, e.g., by means of a system of conveyor belts arranged in a series. The product $y(t)$ is then relayed from one conveyor (\tilde{T}_1) to the next one (\tilde{T}_2) by means of the so-called relay L . The relay, in addition to directing the product flow from conveyor \tilde{T}_1 to conveyor \tilde{T}_2 , by virtue of its capacity absorbs irregularities in the supply of the product $y(t)$, preventing the occurrence of the so-called barrier. We will define a barrier as a random event occurring if the contents of the relay L are equal to its volume V_1 and the level of the product $y(t)$ exceeds the flow capacity d of the relay. The occurrence of a barrier within a certain time interval will cause an interruption in the functioning of conveyors \tilde{T}_1 and \tilde{T}_2 , and consequently a stoppage of the entire transport subsystem \tilde{T} working with subsystems \tilde{P} and M .

The determination of the probability of a barrier occurring in the system under consideration as well as the determination of the mean quantity of the product that spills over from the relay L as a result of the occurrence of such a barrier is of great practical importance in the analysis of the efficiency of an inventory storage and issue system.

The functioning of the relay L may be described by a vector stochastic process $(z(t), y(t), v(t))$, whose coordinates are defined as follows:

- $v(t)$ – a process describing the functioning of the transport subsystem \tilde{T} ;
- $y(t)$ – a process describing the quantity of production generated by the production subsystem \tilde{P} , conveyed by means of the transport subsystem to the receiver E ;
- $z(t)$ – a process describing the filling level of relay L .

Below we will assume that $y(t)$ is a homogeneous, continuous and separable Markov process (cf., e.g., [3]) with a finite number of different states y_1, y_2, \dots, y_n , which are non-negative real numbers, and $\pi_{ij}^{(t)}$ denotes the intensity of the transition

of this process from state y_i to state y_j within time T_b , $l = 1, 2, \dots, m$. The intensities $\pi_{ij}^{(l)}$ will be termed the parameters of the process $y(t)$. In order to simplify the notation we will denote the time period T_l and the set of instants (moments) making up the period T_l by the same symbol.

As is easy to note, the filling process $z(t)$ in the interval $[\alpha_1, \alpha_2)$ of the stability of the process $y(t)$ satisfies the condition

$$z(t) = h[z(t_1) + (y(t_1) - d)(t - t_1)], \quad \alpha_1 \leq t_1 < t < \alpha_2,$$

where

$$h(r) = \begin{cases} 0, & r \leq 0, \\ r, & 0 < r < V_1, \\ V_1, & r \geq V_1. \end{cases}$$

In order to explicitly incorporate the behaviour of the transport subsystem \tilde{T} in the analysis of the functioning of the subsystem L , we will introduce a process $v(t)$ defined by the following formula:

$$v(t) = \begin{cases} 1, & \text{if subsystem } \tilde{T} \text{ is in the working state,} \\ 0, & \text{if subsystem } \tilde{T} \text{ is not working (is in a failure state).} \end{cases}$$

Thus, the filling level $z(t)$ of the relay is controlled by the process $w(t) = y(t)v(t)$, where $y(t)$ describes the quantity of the production of subsystem \tilde{P} relayed to subsystem E . Assume that the processes $y(t)$ and $v(t)$ are independent and that $v(t)$ is a continuous, homogeneous and separable Markov process with intensities $\pi_{10}^* = \pi_1^*$ (the intensity of transition of subsystem \tilde{T} from the working state to the interruption (failure) state), $\pi_{01}^* = \pi_0^*$ (the intensity of transition of subsystem \tilde{T} from the interruption (failure) state to the working state).

In order to also reflect changes over time of the parameters π_{10}^* , π_{01}^* of the controlling process $v(t)$, we will analyse the functioning of subsystem L during m consecutive time periods T_1, T_2, \dots, T_m . Therefore, let the period T_l correspond to intensities $\pi_{10}^{*l} = \pi_1^{*l}$, $\pi_{01}^{*l} = \pi_0^{*l}$ ($l = 1, 2, \dots, m$).

The purpose of [2] and subsequent papers is to obtain a quantitative identification of the barrier to the functioning of an inventory storage and issue system. In paper [2], we analyse the intermediate states of subsystem L defined by a random event of the form: $0 < z(t) < V_1$.

This paper is a continuation of investigations reported in paper [2], where the notion of a barrier in the functioning of an inventory storage and issue system with non-

aggregated dynamic-parameter input is defined. The paper presented the general operating principles of the system and the conditional distributions for intermediate states of the subsystem L . These distributions will be used below to derive a system of equations satisfied by the probability density functions defined in [2].

2. System of equations for the density functions

In order to obtain relations satisfied by the density functions $f_k^{ll}(z, t)$ (cf., equation (1) in [2]), we will use the notations, terminology and results given in [2]. We will also assume that the supply of the product to subsystem M and the process describing the functioning of the transport subsystem are Markov processes (cf. e.g. [3]).

Taking into account equations (1), (12), (18)–(32) from [1], for $x_k > 0$, $t \in T_l$, $t + \tau \in T_l$, $0 < \alpha < V_1$, we get

$$\begin{aligned} Q_k^{ll}(\alpha, t + \tau) &= \int_0^\alpha f_k^{ll}(z, t + \tau) dz \\ &= \sum_i \left\{ \int_0^{V_1} q_{ik}^{lll}(z, \alpha; \tau, t) f_i^{ll}(z, t) dz + Q_i^{ll}(\{V_1\}, t) q_{ik}^{lll}(V_1, \alpha; \tau, t) + Q_i^{ll}(\{0\}, t) q_{ik}^{lll}(0, \alpha; \tau, t) \right\} \\ &\quad + \int_0^{V_1} q_{kk}^{0ll}(z, \alpha; \tau, t) f_k^{0l}(z, t) dz + Q_k^{0l}(\{V_1\}, t) q_{kk}^{0ll}(V_1, \alpha; \tau, t) + Q_k^{0l}(\{0\}, t) q_{kk}^{0ll}(0, \alpha; \tau, t) \\ &= B_{k,l}^{+1} + B_{k,l}^{0l} + B_{k,l}^{-1}, \end{aligned} \quad (1)$$

where

$$\begin{aligned} &B_{k,l}^{-1} \\ &\sum_{\substack{i \neq k \\ x_i < 0}} \left\{ \int_0^{V_1} q_{ik}^{lll}(z, \alpha; \tau, t) f_i^{ll}(z, t) dz + Q_i^{ll}(\{V_1\}, t) q_{ik}^{lll}(V_1, \alpha; \tau, t) + Q_i^{ll}(\{0\}, t) q_{ik}^{lll}(0, \alpha; \tau, t) \right\} \\ &= \sum_{\substack{i \neq k \\ x_i < 0}} \left\{ \int_0^{-\tau x_i} \left[(1 - \pi_1^{*l} \tau) \frac{\pi_{ik}^{(l)} z}{-x_i} + o^{(l)}(\tau; z) \right] f_i^{ll}(z, t) dz + \int_{-\tau x_i}^{\alpha - \tau x_k} [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{ll}(z, t) dz \right. \\ &\quad \left. + \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} \left[(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \left(\tau - \frac{\alpha - z - \tau x_k}{x_i - x_k} \right) + o^{(l)}(\tau; \alpha, z) \right] f_i^{ll}(z, t) dz + \int_{\alpha - \tau x_i}^{V_1} o^{(l)}(\tau) f_i^{ll}(z, t) dz \right\}, \end{aligned} \quad (2)$$

$$\begin{aligned}
 & B_{k,l}^{01} \\
 = & \int_0^{V_1} q_{kk}^{11l}(z, \alpha; \tau, t) f_k^{1l}(z, t) dz + Q_k^{1l}(\{V_1\}, t) q_{kk}^{11l}(V_1, \alpha; \tau, t) + Q_k^{1l}(\{0\}, t) q_{kk}^{11l}(0, \alpha; \tau, t) \\
 + & \int_0^{V_1} q_{kk}^{01l}(z, \alpha; \tau, t) f_k^{0l}(z, t) dz + Q_k^{0l}(\{V_1\}, t) q_{kk}^{01l}(V_1, \alpha; \tau, t) + Q_k^{0l}(\{0\}, t) q_{kk}^{01l}(0, \alpha; \tau, t) \\
 = & \int_0^{\alpha - \tau x_k} [(1 - \pi_1^{*l} \tau)(1 - \pi_k^{(l)} \tau) + o^{(l)}(\tau)] f_k^{1l}(z, t) dz + \int_0^{V_1} o^{(l)}(\tau) f_k^{1l}(z, t) dz \\
 & + Q_k^{1l}(\{V_1\}, t) o^{(l)}(\tau) + Q_k^{1l}(\{0\}, t) [(1 - \pi_1^{*l} \tau)(1 - \pi_k^{(l)} \tau) + o^{(l)}(\tau)] \\
 & \int_0^{\tau d} \left[(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \frac{z}{a} f_k^{0l}(z, t) + o^{(l)}(\tau, z) \right] f_k^{0l}(z, t) dz \\
 + & \int_{\tau d}^{\alpha - \tau x_k} [(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau)] f_k^{0l}(z, t) dz + \int_{\alpha + \tau d}^{V_1} o^{(l)}(\tau) f_k^{0l}(z, t) dz \\
 + & \int_{\alpha - \tau x_k}^{\alpha + \tau d} \left[(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \left(\tau + \frac{\alpha - z - \tau x_k}{d + x_k} \right) + o^{(l)}(\tau; \alpha, z) \right] f_k^{0l}(z, t) dz \\
 & + Q_k^{0l}(\{V_1\}, t) o^{(l)}(\tau; z) + Q_k^{0l}(\{0\}, t) o^{(l)}(\tau),
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 & B_{k,l}^{+1} \\
 = & \sum_{\substack{i \neq k \\ x_i \geq 0}} \left\{ \int_0^{V_1} q_{ik}^{11l}(z, \alpha; \tau, t) f_i^{1l}(z, t) dz + Q_i^{1l}(\{V_1\}, t) q_{ik}^{11l}(V_1, \alpha; \tau, t) + Q_i^{1l}(\{0\}, t) q_{ik}^{11l}(0, \alpha; \tau, t) \right\} \\
 = & \sum_{\substack{i \neq k \\ x_i > x_k}} \left\{ \int_0^{\alpha - \tau x_i} [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{1l}(z, t) dz \right. \\
 & \left. + \int_{\alpha - \tau x_i}^{\alpha - \tau x_k} \left[(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{\alpha - z - \tau x_k}{x_i - x_k} + o^{(l)}(\tau; \alpha, z) \right] f_i^{1l}(z, t) dz \right. \\
 + & \left. \int_{\alpha - \tau x_k}^{V_1} o^{(l)}(\tau) f_i^{1l}(z, t) dz + [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] Q_i^{1l}(\{0\}, t) + o^{(l)}(\tau) Q_i^{1l}(\{V_1\}, t) \right\} \\
 & + \sum_{\substack{i \neq k \\ x_i < x_k}} \left\{ \int_0^{\alpha - \tau x_k} [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{1l}(z, t) dz \right. \\
 + & \left. \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} \left[(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \left(\tau - \frac{\alpha - z - \tau x_k}{x_i - x_k} \right) + o^{(l)}(\tau; \alpha, z) \right] f_i^{1l}(z, t) dz + o^{(l)}(\tau; \alpha, z) \right. \\
 + & \left. \int_{\alpha - \tau x_i}^{V_1} o^{(l)}(\tau) f_i^{1l}(z, t) dz + [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] Q_i^{1l}(\{0\}, t) + o^{(l)}(\tau) Q_i^{1l}(\{V_1\}, t) \right\}.
 \end{aligned} \tag{4}$$

By differentiating both sides of equation (1) with respect to α , taking into account relations (2)–(4), we get

$$f_k^{ll}(\alpha, t + \tau) = \frac{d}{d\alpha} B_{k,l}^{+1} + \frac{d}{d\alpha} B_{k,l}^{0l} + \frac{d}{d\alpha} B_{k,l}^{-1}, \quad (5)$$

where

$$\begin{aligned} \frac{d}{d\alpha} B_{k,l}^{-1} = & \sum_{\substack{i \neq k \\ x_i < 0}} \left\{ [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{ll}(\alpha - \tau x_i, t) - o^{(l)}(\tau) f_i^{ll}(\alpha - \tau x_i, t) \right. \\ & \left. + \frac{d}{d\alpha} \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} o^{(l)}(\tau; \alpha, z) f_i^{ll}(z, t) dz - \frac{1 - \pi_1^{*l} \tau}{x_i - x_k} \pi_{ik}^{(l)} \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} f_i^{ll}(z, t) dz - (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau f_i^{ll}(\alpha - \tau x_k, t) \right\}, \end{aligned}$$

$$\begin{aligned} \frac{d}{d\alpha} B_{k,l}^{0l} = & [(1 - \pi_1^{*l} \tau)(1 - \pi_k^{(l)}) + o^{(l)}(\tau)] f_k^{ll}(\alpha - \tau x_k, t) - o^{(l)}(\tau) f_k^{ll}(\alpha - \tau x_k, t) \\ & + [(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau)] f_k^{0l}(\alpha - \tau x_k, t), \end{aligned}$$

$$\begin{aligned} \frac{d}{d\alpha} B_{k,l}^{+1} = & \sum_{\substack{i \neq k \\ x_i > x_k}} \left\{ \frac{1 - \pi_1^{*l} \tau}{x_i - x_k} \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} f_i^{ll}(z, t) dz + [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{ll}(\alpha - \tau x_i, t) \right. \\ & \left. + \frac{d}{d\alpha} \int_{\alpha - \tau x_i}^{\alpha - \tau x_k} o^{(l)}(\tau; \alpha, z) f_i^{ll}(z, t) dz - (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau f_i^{ll}(\alpha - \tau x_i, t) - o^{(l)}(\tau) f_i^{ll}(\alpha - \tau x_k, t) \right\} \\ & \sum_{\substack{i \neq k \\ x_i < x_k}} \left\{ \frac{1 - \pi_1^{*l} \tau}{x_k - x_i} \pi_{ik}^{(l)} \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} f_i^{ll}(z, t) dz + [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_k^{ll}(\alpha - \tau x_k, t) \right. \\ & \left. + \frac{d}{d\alpha} \int_{\alpha - \tau x_k}^{\alpha - \tau x_i} o^{(l)}(\tau; \alpha, z) f_i^{ll}(z, t) dz - (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau f_i^{ll}(\alpha - \tau x_k, t) - o^{(l)}(\tau) f_i^{ll}(\alpha - \tau x_i, t) \right\}. \end{aligned}$$

Lets us expand the functions $f_k^{ll}(\alpha - \tau x_k, t)$, $f_k^{0l}(\alpha - \tau x_k, t)$, which occur in (5), using the Taylor formula into a second degree term, move the expression $f_k^{ll}(\alpha, t)$ to the left-hand side of equation (5), divide both sides of the obtained equation by τ , and go to the limit for $\tau \rightarrow 0$. As a result of these operations, we obtain

$$\begin{aligned} \frac{\partial f^{ll}(\alpha, t)}{\partial t} = & -x_k \frac{\partial f_k^{ll}(\alpha, t)}{\partial \alpha} - (\pi_k^{(l)} + \pi_1^{*l}) f_k^{ll}(\alpha, t) + \sum_{i \neq k} \pi_{ik}^{(l)} f_i^{ll}(\alpha, t) + \pi_0^{*l} f_k^{0l}(\alpha, t), \quad (6) \\ & 0 < \alpha < V_1, t \in T_1, k = 1, 2, \dots, n. \end{aligned}$$

Relation (6) is derived similarly for $x_k \leq 0$. Relations satisfied by the density functions $f_k^{0l}(\alpha, t)$ can be obtained analogously. Thus, the density functions $f_k^{ll}(z, t)$, $f_k^{0l}(z, t)$ satisfy the following system of differential equations:

$$\begin{aligned} \frac{\partial f_k^{ll}(z, t)}{\partial t} &= -x_k \frac{\partial f_k^{ll}(z, t)}{\partial z} - (\pi_k^{(l)} + \pi_1^{*l}) f_k^{ll}(z, t) + \sum_{i \neq k} \pi_{ik}^{(l)} f_i^{ll}(z, t) + \pi_0^{*l} f_k^{0l}(z, t), \\ &0 < z < V_1, t \in T_l, k = 1, 2, \dots, n, \\ \frac{\partial f_k^{0l}(z, t)}{\partial t} &= a \frac{\partial f_k^{0l}(z, t)}{\partial z} - (\pi_k^{(l)} + \pi_0^{*l}) f_k^{0l}(z, t) + \sum_{i \neq k} \pi_{ik}^{(l)} f_i^{0l}(z, t) + \pi_1^{*l} f_k^{ll}(z, t), \\ &0 < z < V_1, t \in T_l, k = 1, 2, \dots, n. \end{aligned} \quad (7)$$

The relations in (7) will be used in the authors' further work for the identification of a barrier in the functioning of the system under consideration.

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Funkcje gęstości procesu w zagadnieniu identyfikacji bariery funkcjonowania pewnego systemu gromadzenia i wydawania zapasów

Przedmiotem badania jest bariera działania pewnego systemu gospodarki zasobami w przypadku, gdy wejście magazynu–zbiornika jest procesem niezagregowanym o dynamicznych parametrach. Wprowadzono układ równań różniczkowych, który spełniają funkcje gęstości rozkładów prawdopodobieństwa stanów pośrednich podsystemu L . Układ ten wyraża związki między gęstościami i parametrami funkcjonowania podsystemu transportowego i parametrami procesu podaży produktu.

Słowa kluczowe: *zapasy, bariera, transport, układ równań różniczkowych*