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DETERMINING PREFERENCE ORDERS BASED ON A MAJORITY MARGIN MATRIX

When determining a group ranking, the information about experts' opinions may be sometimes incomplete. In such cases, usually only the outranking matrix L or the majority margin matrix ΔL is available. Debord [2] presented a technique for constructing the set of experts' opinions based on the majority margin matrix. The method proposed – simple and efficient – provides a set of experts' opinions consistent with the given majority margin matrix.

Key words: *profile of preference orders, pairwise comparisons, outranking matrix, majority margin matrix, Debord's method*

1. Introduction

There are several methods of aggregating experts' opinions into a group ranking. Some of them use pairwise comparisons and the corresponding outranking matrix [4]. For each pair of alternatives evaluated O_i, O_j, $i, j = 1, \ldots, n$, let l_{ij} denote the number of experts who regarded alternative O_i to be better – in the sense of the criterion adopted – than alternative O_j and l_{ji} denotes the number of experts who presented the opposite opinion. When tied alternatives are allowed in experts' opinions, the following formula holds

$$l_{ij} + l_{ji} + m_{ij} = K \tag{1}$$

where m_{ij} denotes the number of experts who regarded O_i and O_j as being tied, in the sense of the criterion assumed, and K denotes the number of experts.

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If tied alternatives can occur in experts' opinions, then generally it is assumed – for Eq. (1) to hold – that each of tied alternatives receives a half of an expert's vote. Hence, the outranking matrix is composed of elements l_{ij} and l_{ji} (when there are no ties) or elements $l_{ij} + m_{ij}/2$ and $l_{ji} + m_{ij}/2$ (when ties occur). Thus the outranking matrix $L = [l_{ij}]$ indirectly contains information about the total number of experts.

In some situations, the outranking matrix may be unavailable, but the majority margin matrix $\Delta L = L - L^T$ [2, 3] is available. It is worth noticing that the form of the difference Δl_{ij} does not depend on whether ties occur or not. Thus, knowledge of the majority margin matrix ΔL is insufficient to unambiguously state whether there were tied alternatives in experts' opinions or not.

Difficulties in establishing the real (in the sense of the outranking matrix) experts' opinions and the need for using the majority margin matrix instead may be encountered in some real-life tasks of determining a group decision. For example, one can refer to the following problems:

- One of the methods of determining a group decision consists of introducing a two-stage procedure. In the first stage, experts provide their opinions and some additional data concerning their competence in the area considered or arguments leading to their evaluation (e.g. theoretical analysis, personal experience, profound knowledge of the results of research, intuition). In the second stage, this information may result in excluding some opinions or in asking for additional explanation. If this is the case, further investigation will rely on the majority margin matrix only.
- A similar situation occurs when feed-back evaluation is performed, i.e. the experts' evaluation is performed over several rounds and the results of each round are discussed by the experts (to an extent acceptable by the method). As a result of such a procedure, the outranking matrix may be subject to changes. Hence, again the further process of determining a group ranking will be limited to considering the majority margin matrix.
- In the case of complex evaluation by experts, particularly concerning the distant future, even carefully selected experts may acknowledge that they are not able to compare some alternatives to other ones. It may also happen, when the number of alternatives is substantial, that some experts' opinions may be intransitive. If this is the case, it should be taken into account. A possible solution is to limit the data to the majority margin matrix.
- Some methods of determining a group decision use only the majority margin matrix, e.g. Tideman's ranked pairs method [7] or Schulze's method [6].

In the problems mentioned, the group decision may be determined on the basis of the majority margin matrix alone.

If, for some reasons, one would like to compare the group decision determined based on a majority margin matrix with another one determined using methods which rely on outranking matrices (e.g. the Condorcet or Borda method), the lack of the corresponding outranking matrix would be an obstacle. Debord [2] proposed a simple method of constructing a set of experts' preference orders (as well as the corresponding outranking matrix) such that the majority margin matrix obtained is consistent with the majority matrix given. In both cases, the group ranking is the same.

Lamboray (e.g. [3]) applied Debord's method to obtain a set of prudent orders, which was next analyzed and compared with other procedures of ranking aggregation. Due to its simplicity, this method can be easily applied. In this paper, a description of this method and its applications are presented.

2. Definitions

Assume that there is a set of n alternatives $\mathcal{O} = \{O_1, O_2, ..., O_n\}$ and a set of K experts who evaluate the alternatives. One assumes that the experts' opinions are given in an order scale. Moreover, one assumes that all the alternatives are compared and ties between alternatives may occur in experts' opinions.

Let us denote:

$O_i \succ^k O_j$, if the k-th expert regards alternative O_i as better – in the sense of the criterion (the set of criteria) adopted than alternative O_j,

$O_i \approx^k O_j$, if the k-th expert regards alternatives O_i and O_j as tied in the sense of the criterion (the set of criteria) adopted,

$O_i \prec^k O_j$, if the k-th expert regards alternative O_j as better in the sense of the criterion (the set of criteria) adopted than alternative O_i.

The k-th expert opinion may be given in the form of a pairwise comparisons matrix $A^k = [a_{ij}^k]$

$$a_{ij}^k = \begin{cases} 1 & \text{if } O_i \succ^k O_j \\ 0 & \text{if } O_i \approx^k O_j \\ -1 & \text{if } O_i \prec^k O_j \end{cases} \tag{2}$$

It is assumed that all the relations (between alternatives) considered are defined with respect to a given criterion (set of criteria). For simplicity, in further considerations this assumption will be omitted.

In problems of determining a group ranking it is often assumed that the information about experts' opinions is given in the form of an outranking matrix L only.

$$L = \begin{array}{c|cccc} & O_1 & O_2 & \cdots & O_n \\ \hline O_1 & - & l_{12} & \cdots & l_{1n} \\ O_2 & l_{21} & - & \cdots & l_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_n & l_{n1} & l_{n2} & \cdots & - \end{array} \tag{3}$$

The majority margin matrix is defined as follows:

$$\Delta L = L - L^T \tag{4}$$

If there are no ties between alternatives in the expert opinions, then from (1)

$$\Delta l_{ij} = l_{ij} - l_{ji} = 2l_{ij} - K \tag{5}$$

It is worth noting that Δl_{ij} is even (odd) when K is even (odd).

For the case when ties between alternatives can occur in experts' opinions, one has

$$\tilde{l}_{ij} = l_{ij} + 0.5m_{ij} \tag{6}$$

and

$$\Delta l_{ij} = \tilde{l}_{ij} - \tilde{l}_{ji} = 2l_{ij} + m_{ij} - K = 2\tilde{l}_{ij} - K \tag{7}$$

$2\tilde{l}_{ij}$ may take even or odd values, thus Δl_{ij} may be even or odd as well.

Assume that experts present their opinions in the form of preference orders P^t ($t = 1, ..., T$) of elements of an n alternative set, where T is the number of all the preference orders of elements from this set, e.g. $P^t = \{O_{i_1}, O_{i_2}, \ldots, (O_{i_k}, O_{i_l}), \ldots, O_{i_n}\}$, tied alternatives are given i brackets.

Definition 1 [5]

A profile of preference orders for a set of n alternatives is a vector

$$p = (p_1, \ ..., \ p_t, \ ..., \ p_T) \tag{8}$$

where p_t denotes the number of experts whose opinion is of the form P^t ($t = 1, ..., T$).

If the number of experts considered is equal to K, then

$$\sum_{t=1}^{T} p_t = K \tag{9}$$

Such a profile contains – given the assumptions adopted – the whole information about the problem to be solved.

If there are no ties between alternatives in the experts' opinions, then $T = n!$. Otherwise, the number of all the possible preference orders for a set of n alternatives may be approximated using $T \approx n!/2(\log 2)^{n+1}$ [1].

It is worth noting that some authors (e.g. [2]) define the profile as a set of preference orders. It can be shown that both formulations coincide.

3. The Debord method

Assume that there are no ties between alternatives in the experts' opinions. Let us denote by $O \ldots ij$ any permutation of alternatives from the set $\mathcal{O} \backslash \{O_i, O_j\}$ and by $-O \ldots ij$ an inverse permutation.

For each pair of alternatives (O_i, O_j) let us consider two preference orders

$$O \ldots ij, O_i, O_j \tag{10}$$

and

$$O_i, O_j, -O \ldots ij \tag{11}$$

Debord noticed that for such a pair of preference orders the outranking matrix $L^{(ij)}$ and the majority margin matrix $\Delta L^{(ij)}$ are of the form

$$L^{(ij)} = [l_{ij}^{(ij)}] = \begin{array}{c|ccccccc} & O_1 & \cdots & O_i & \cdots & O_j & \cdots & O_n \\ \hline O_1 & - & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O_i & 1 & \vdots & - & \cdots & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ O_j & 1 & \vdots & 0 & \vdots & - & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ O_n & 1 & \vdots & 1 & \vdots & 1 & \vdots & - \end{array} \tag{12}$$

and

$$\Delta L^{(ij)} = [\Delta l_{ij}^{(ij)}] = \begin{array}{c|ccccccc} & O_1 & \dots & O_i & \dots & O_j & \dots & O_n \\ \hline O_1 & - & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \dots & \dots & \dots & \dots \\ O_i & 0 & \vdots & - & \dots & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \dots & \dots \\ O_j & 0 & \vdots & -2 & \vdots & - & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \dots \\ O_n & 0 & \vdots & 0 & \vdots & 0 & \vdots & - \end{array} \tag{13}$$

$\Delta L^{(ij)}$ is a null matrix except for the elements (i, j) and (j, i), which are equal to 2 or –2, respectively.

The Debord method consists of determining a profile of linear preference orders generated from pairs of orders of the form (10) and (11), for which its majority margin matrix $\Delta L'$ is consistent with a given matrix ΔL. (A linear preference order is a binary relation on the set of alternatives which is complete, transitive and antisymmetric, see e.g. [3].) Debord proved the following lemma.

Lemma ([2], [3])

Assume an $n \times n$ matrix ΔL, with even elements Δl_{ij}, is given. Then there exists a profile p_D of preference orders for n alternatives such that ΔL is equal to its majority margin matrix. This profile is constructed as follows:

For each $\Delta l_{ij} > 0$ one forms a pair of preference orders

$$O \dots ij,\ O_i,\ O_j \tag{14}$$

and

$$O_i,\ O_j,\ -O \dots ij \tag{15}$$

which appears in the profile under construction $\Delta l_{ij}/2$ times.

The elements of the majority margin matrix $\Delta L'$ of such a profile are given as follows:

$$\Delta l'_{ij} = \Delta l_{ij}^{(ij)} \left. \frac{\Delta l_{ij}}{2} \right|_{\Delta l_{ij} > 0} = 2 \frac{\Delta l_{ij}}{2} = \Delta l_{ij} \tag{16}$$

Hence, $\Delta L' = \Delta L$.

The elements of the majority margin matrix are required to be even, thus one has to analyze the following situations:

a) the number of experts K is even, hence $\Delta l_{ij} = 2l_{ij} - K$ is even.

b) the number of experts K is odd, hence $\Delta l_{ij} = 2l_{ij} - K$ is odd. In such a case, the coefficients of the majority margin matrix should be multiplied by 2, $\overline{\Delta l}_{ij} = 2(2l_{ij} - K)$.

4. Analysis of the number of preference orders derived using the Debord method

Let us define the following sets of indices

$$I_1 = \{(i, j, j > i) : \Delta l_{ij} > 0\} \tag{17}$$

$$I_2 = \{(i, j, j > i) : \Delta l_{ji} > 0\} \tag{18}$$

and

$$I_3 = \{(i, j, \; i \neq j) : \Delta l_{ij} = 0\} \tag{19}$$

According to the methodology adopted, the number of preference orders K' that results from applying the Debord method is equal to

$$K' = \sum_{(i,j)\in I_1} \Delta l_{ij} + \sum_{(i,j)\in I_2} \Delta l_{ji} \tag{20}$$

One has

$$K' = \sum_{(i,j)\in I_1} \Delta l_{ij} + \sum_{(i,j)\in I_2} \Delta l_{ji} = 2\sum_{(i,j)\in I_1} l_{ij} - \sum_{(i,j)\in I_1} K + 2\sum_{(i,j)\in I_2} l_{ji} - \sum_{(i,j)\in I_2} K \tag{21}$$

If $I_3 = \varnothing$, then the number of pairs of alternatives considered is equal to $n(n-1)/2$. Hence,

$$\sum_{(i,j)\in I_1} \Delta l_{ij} + \sum_{(i,j)\in I_2} \Delta l_{ji} = 2\left(\sum_{(i,j)\in I_1} l_{ij} + \sum_{(i,j)\in I_2} l_{ji}\right) - \frac{n(n-1)}{2} K \tag{22}$$

The condition $\Delta l_{ij} > 0$ implies that $l_{ij} > K/2$. Likewise, $\Delta l_{ji} > 0$ implies that $l_{ji} > K/2$. Thus

$$\sum_{(i,j)\in I_1} l_{ij} + \sum_{(i,j)\in I_2} l_{ji} > \frac{n(n-1)}{2}\frac{K}{2} \tag{23}$$

One may assume that

$$\sum_{(i,j)\in I_1} l_{ij} + \sum_{(i,j)\in I_2} l_{ji} = \left(\frac{n(n-1)}{2} + a\right)\frac{K}{2}, \qquad a > 0 \tag{24}$$

It follows from Eqs. (22) and (24) that

$$K' = \sum_{(i,j)\in I_1} \Delta l_{ij} + \sum_{(i,j)\in I_2} \Delta l_{ji} = aK \tag{25}$$

For $\Delta l_{ij} = 2, \quad i,j \in I_1$ and $\Delta l_{ji} = 2, \quad i,j \in I_2$, one has

$$K'_{\min} = 2\frac{n(n-1)}{2} = n(n-1) \tag{26}$$

Likewise, for $\Delta l_{ij} = K, \quad (i,j) \in I_1$ and $\Delta l_{ji} = K, \quad (i,j) \in I_2$, one has

$$K'_{\max} = \frac{n(n-1)}{2}K \tag{27}$$

so

$$\frac{K'_{\max}}{K'_{\min}} = \frac{n(n-1)}{n(n-1)} \times \frac{K}{2} = \frac{K}{2} \tag{28}$$

One also has

$$K'_{\max} - K'_{\min} = n(n-1)\frac{K}{2} - n(n-1) = n(n-1)\left(\frac{K}{2} - 1\right) = n(n-1)\left(\frac{K-2}{2}\right) \tag{29}$$

It follows from Eq. (25) that

$$K' - K = K(a-1) \tag{30}$$

so

$$l'_{ij} = l_{ij} + \frac{K'-K}{2} = l_{ij} + \frac{K(a-1)}{2} = l_{ij} + (a-1)\frac{K}{2} \tag{31}$$

If (to ensure evenness) the ΔL matrix was multiplied by 2, then – when applying Eq. (31) – l_{ij} and K should also be multiplied by 2.

It is worth noticing that

$$K'_{\min} - K = \left(\frac{n(n-1)}{K} - 1\right)K \tag{32}$$

It follows that for

$$n(n-1) < K, \quad K'_{\min} - K < 0 \quad \text{and} \quad l'_{ij} < l_{ij} \tag{33}$$

$$n(n-1) = K, \quad K'_{\min} - K = 0 \quad \text{and} \quad l'_{ij} = l_{ij} \tag{34}$$

$$n(n-1) > K, \quad K'_{\min} - K > 0 \quad \text{and} \quad l'_{ij} > l_{ij} \tag{35}$$

One also has

n	3	4	5	6	7	8	9	10
$K'_{\min} = n(n-1)$	6	12	20	30	42	56	72	90
$K'_{\max} = K'_{\min}K/2$	$3K$	$6K$	$10K$	$15K$	$21K$	$28K$	$36K$	$45K$

If $I_3 \neq \varnothing$, then $K'_{\min} = n(n-1) - s/2$, where s is the cardinality of the set I_3.

It is worth emphasizing that if all the l_{ij} or l_{ji} are equal to K, then $a = n(n-1)/2$. One has $K'_{\min} \le K' \le K'_{\max}$, hence

$$\frac{n(n-1)}{K} \le a \le \frac{n(n-1)}{2} \tag{36}$$

For the number of preference orders determined using the Debord method based on the outranking matrix, to be a multiple of K, a should be a natural number. For $a = 1$, both numbers are the same.

5. Examples

Example 1 [3]

In Table 1, the preference orders for four alternatives given by 46 experts are presented.

Table 1. Preference orders for four alternatives given by 46 experts

Number of occurrences	Preference order
8	O_1, O_2, O_3, O_4
4	O_4, O_3, O_1, O_2
9	O_2, O_3, O_4, O_1
6	O_1, O_4, O_2, O_3
4	O_3, O_4, O_2, O_1
3	O_3, O_1, O_2, O_4
6	O_4, O_2, O_3, O_1
3	O_4, O_1, O_3, O_2
3	O_1, O_3, O_4, O_2

The L and ΔL matrices are as follows.

$$L = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 27 & 20 & 20 \\ O & 19 & - & 29 & 20 \\ O_3 & 26 & 17 & - & 27 \\ O_4 & 26 & 26 & 19 & - \end{array} \qquad \Delta L = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 8 & -6 & -6 \\ O_2 & -8 & - & 12 & -6 \\ O_3 & 6 & -12 & - & 8 \\ O_4 & 6 & 6 & -8 & - \end{array} \tag{37}$$

According to the Debord method, 6 pairs of preference orders are to be considered. They are presented in Table 2. (O_i, O_j pairs for which $\Delta l_{ij} > 0$ are shaded.)

For the profile p_D of preference orders obtained, the outranking matrix L' and majority margin matrix $\Delta L'$ are consistent with Eq. (37).

The number of nonzero preference orders in the p_D profile can be evaluated as follows:

$$I_1 = \{(1, 2), (2, 3), (3, 4)\} \tag{38}$$

$$I_2 = \{(3, 1), (4, 1), (4, 2)\} \tag{39}$$

Table 2. Pairs of preference orders to be considered

Number of occurrences	Pair of preference orders
4	O_1, O_2, O_3, O_4 O_4, O_3, O_1, O_2
6	O_2, O_3, O_4, O_1 O_1, O_4, O_2, O_3
3	O_3, O_1, O_2, O_4 O_4, O_2, O_3, O_1
4	O_3, O_4, O_2, O_1 O_1, O_2, O_3, O_4
3	O_4, O_1, O_3, O_2 O_2, O_3, O_4, O_1
3	O_4, O_2, O_3, O_1 O_1, O_3, O_4, O_2
$K' = 2\times 23 = 46$	

$$\sum_{(i,j)\in I_1} l_{ij} = 27 + 29 + 27 = 83, \qquad \sum_{(i,j)\in I_2} l_{ji} = 26 + 26 + 26 = 78 \tag{40}$$

thus

$$\sum_{(i,j)\in I_1} l_{ij} + \sum_{(i,j)\in I_2} l_{ji} = 161 = \left(\frac{4\times 3}{2} + a\right)23, \text{ and } a = 1 \tag{41}$$

Example 2

Assume the following preference orders for four alternatives given by five experts:

$$\begin{aligned} &P^1: O_1, O_2, O_4, O_3 \\ &P^2: O_4, O_1, O_3, O_2 \\ &P^3: O_4, O_3, O_1, O_2 \\ &P^4: O_1, O_2, O_3, O_4 \\ &P^5: O_4, O_3, O_1, O_2 \end{aligned} \tag{42}$$

The outranking matrix L and majority margin matrix ΔL are of the form:

$$L = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 5 & 3 & 2 \\ O_2 & 0 & - & 2 & 2 \\ O_3 & 2 & 3 & - & 1 \\ O_4 & 3 & 3 & 4 & - \end{array} \qquad \Delta L = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 5 & 1 & -1 \\ O_2 & -5 & - & -1 & -1 \\ O_3 & -1 & 1 & - & -3 \\ O_4 & 1 & 1 & 3 & - \end{array} \tag{43}$$

To ensure the evenness of the majority margin matrix, it should be multiplied by 2:

$$\overline{\Delta L} = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 10 & 2 & -2 \\ O_2 & -10 & - & -2 & -2 \\ O_3 & -2 & 2 & - & -6 \\ O_4 & 2 & 2 & 6 & - \end{array} \tag{44}$$

Hence, in the profile p_D determined by the Debord method, i.e. on the basis of the majority margin matrix only, the following preference orders should be taken into account:

Table 3. Preference orders determined by the Debord method

Number of occurrences	Pair of preference orders
5	O_1, O_2, O_4, O_3 O_3, O_4, O_1, O_2
1	O_1, O_3, O_4, O_2 O_2, O_4, O_1, O_3
1	O_3, O_2, O_4, O_1 O_1, O_4, O_3, O_2
1	O_4, O_1, O_3, O_2 O_2, O_3, O_4, O_1
1	O_4, O_2, O_3, O_1 O_1, O_3, O_4, O_2
3	O_4, O_3, O_2, O_1 O_1, O_2, O_4, O_3

For the profile p_D, the outranking matrix L' and majority margin matrix $\Delta L'$ are as follows:

$$L' = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 17 & 13 & 11 \\ O_2 & 7 & - & 11 & 11 \\ O_3 & 11 & 13 & - & 9 \\ O_4 & 13 & 13 & 15 & - \end{array} \qquad \Delta L' = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 10 & 2 & -2 \\ O_2 & -10 & - & -2 & -2 \\ O_3 & -2 & 2 & - & -6 \\ O_4 & 2 & 2 & 6 & - \end{array} \tag{45}$$

It can be seen that the $\overline{\Delta L}$ and $\Delta L'$ matrices are consistent.

Example 3

Assume that the outranking matrix L and the corresponding ΔL matrix are given as follows ($n = 5$, $K = 10$):

$$L = \begin{array}{c|ccccc} & O_1 & O_2 & O_3 & O_4 & O_5 \\ \hline O_1 & - & 4 & 3 & 4 & 6 \\ O_2 & 6 & - & 3 & 4 & 4 \\ O_3 & 7 & 7 & - & 4 & 7 \\ O_4 & 6 & 6 & 6 & - & 4 \\ O_5 & 4 & 6 & 3 & 6 & - \end{array} \qquad \Delta L = \begin{array}{c|ccccc} & O_1 & O_2 & O_3 & O_4 & O_5 \\ \hline O_1 & - & -2 & -4 & -2 & 2 \\ O_2 & 2 & - & -4 & -2 & -2 \\ O_3 & 4 & 4 & - & -2 & 4 \\ O_4 & 2 & 2 & 2 & - & -2 \\ O_5 & -2 & 2 & -4 & 2 & - \end{array} \tag{46}$$

The number of preference orders determined using the Debord method can be evaluated as follows:

$$I_1 = \{(1, 5), (3, 5)\} \tag{47}$$

$$I_2 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 2), (5, 4)\} \tag{48}$$

$$\sum_{(i,j)\in I_1} l_{ij} = 13, \qquad \sum_{(i,j)\in I_2} l_{ji} = 50 \tag{49}$$

$$\sum_{(i,j)\in I_1} l_{ij} + \sum_{(i,j)\in I_2} l_{ji} = \left[\left(\frac{n(n-1)}{2} + a\right)\right]\frac{K}{2} = (10 + a)5 = 63, \qquad K' = 26, \; a = 2.6 \tag{50}$$

Example 4

Assume the following majority margin matrix ΔL is given:

$$\Delta L = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 1 & 8 & 10 \\ O_2 & -1 & - & -2 & 4 \\ O_3 & -8 & 2 & - & 4 \\ O_4 & -10 & -4 & -4 & - \end{array} \qquad \overline{\Delta L} = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 2 & 16 & 20 \\ O_2 & -2 & - & -4 & 8 \\ O_3 & -16 & 4 & - & 8 \\ O_4 & -20 & -8 & -8 & - \end{array} \tag{51}$$

To ensure the evenness of the majority margin matrix, ΔL was multiplied by 2, $\overline{\Delta L} = 2\Delta L$.

In the profile p_D determined using the Debord method, the following preference orders should be taken into account:

Table 4. Preference orders determined by the Debord method

Number of occurrences	Pair of preference orders
1	O_1, O_2, O_3, O_4 O_4, O_3, O_1, O_2
8	O_1, O_3, O_2, O_4 O_4, O_2, O_1, O_3
10	O_1, O_4, O_2, O_3 O_3, O_2, O_1, O_4
4	O_2, O_4, O_1, O_3 O_3, O_1, O_2, O_4
2	O_3, O_2, O_1, O_4 O_4, O_1, O_3, O_2
4	O_3, O_4, O_1, O_2 O_2, O_1, O_3, O_4
$K' = 29\times2 = 58$	

In the paper cited [2], Debord also considered the situation where there were ties between alternatives in the experts' opinions. If this is the case, then in Eqs. (10) and (11) one pair of alternatives O_i, O_j should be replaced by a pair of equally preferred alternatives (O_i, O_j).

Let us consider the following two preference orders

$$O \ldots ij, O_i, O_j \tag{52}$$

and

$$(O_i, O_j), - O \ldots ij \tag{53}$$

For this pair of preference orders, the outranking matrix $L^{(ij)}$ and the majority margin matrix $\Delta L^{(ij)}$ are as follows:

$$L^{(ij)} = \begin{array}{c|ccccccc} & O_1 & \ldots & O_i & \ldots & O_j & \ldots & O_n \\ \hline O_1 & - & \ldots & 1 & \ldots & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ldots & \ldots & \ldots \\ O_i & 1 & \vdots & - & \ldots & 1.5 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ldots & \ldots & \ldots \\ O_j & 1 & \vdots & 0.5 & \vdots & - & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ldots \\ O_n & 1 & \vdots & 1 & \vdots & 1 & \vdots & - \end{array} \qquad \Delta L^{(ij)} = \begin{array}{c|ccccccc} & O_1 & \ldots & O_i & \ldots & O_j & \ldots & O_n \\ \hline O_1 & - & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ldots & \ldots & \ldots \\ O_i & 0 & \vdots & - & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ldots & \ldots & \ldots \\ O_j & 0 & \vdots & -1 & \vdots & - & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ldots \\ O_n & 0 & \vdots & 0 & \vdots & 0 & \vdots & - \end{array} \tag{54}$$

If there are odd elements in the ΔL matrix, taking into account ties between alternatives (Eqs. (52) and (53)) in the profile p_D under investigation – if possible with respect to the nature of the problem – makes it possible to decrease K' – the number of preference orders reconstructed. In such a case, in the profile p''_D determined using the Debord method, the following preference orders should be taken into account (Table 5):

Table 5. Preference orders
in the profile p''_D to be taken into account

Number of occurrences	Pairs of preference orders
1	O_1, O_2, O_3, O_4 $O_4, O_3, (O_1, O_2)$
4	O_1, O_3, O_2, O_4 O_4, O_2, O_1, O_3
5	O_1, O_4, O_2, O_3 O_3, O_2, O_1, O_4
2	O_2, O_4, O_1, O_3 O_3, O_1, O_2, O_4
1	O_3, O_2, O_1, O_4 O_4, O_1, O_3, O_2
2	O_3, O_4, O_1, O_2 O_2, O_1, O_3, O_4

The outranking matrix L'' and the majority margin matrix $\Delta L''$ are as follows:

$$L'' = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 15.5 & 19 & 20 \\ O_2 & 14.5 & - & 14 & 17 \\ O_3 & 11 & 16 & - & 17 \\ O_4 & 10 & 13 & 13 & - \end{array} \qquad \overline{\Delta L''} = \begin{array}{c|cccc} & O_1 & O_2 & O_3 & O_4 \\ \hline O_1 & - & 1 & 8 & 10 \\ O_2 & -1 & - & -2 & 4 \\ O_3 & -8 & 2 & - & 4 \\ O_4 & -10 & -4 & -4 & - \end{array} \tag{55}$$

$K'' = 15 \times 2 = 30.$

If tied alternatives are allowed in the profile, the number of preference orders reconstructed may be significantly diminished. However, taking into account tied alternatives in the profile determined using the Debord method needs further investigation.

6. Concluding remarks

The method described makes it possible to reconstruct experts' opinions (in the sense of an outranking matrix), as well as to determine a group ranking based on incomplete information restricted to a majority margin matrix only. This method of generating preference orders is simple and successful, even for a large number of alternatives, and it requires only some simple arithmetic calculations. It is worth noticing that the solution – a p_D profile – is ambiguous, due to the free choice of the initial permutation $O \ldots ij$.

Moreover, the number of preference orders obtained by the Debord method is generally larger than the number of original orders ($K' > K$).

The profile p_D obtained with the use of the Debord method contains at most $n(n - 1)/2$ pairs of non-zero components and the number of preference orders (see Eq. (20)) – provided there are no ties between alternatives in the experts' opinions – is equal to

$$K' = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left|\Delta l_{ij}\right|$$

In section 4, it was shown how the number of preference orders K' reconstructed by means of the Debord method may vary depending on the real number of preference orders given by K experts, as well as the structure of the majority margin matrix that influences the value of the a coefficient.

In some situations, as mentioned in the Introduction, one neither has knowledge about the number of experts K nor the value of the a coefficient. However, from Eq. (31) it follows that

$$l'_{ij} = l_{ij} + \frac{K' - K}{2}$$

This means that the elements of the outranking matrix reconstructed differ from the elements of the real (unknown) outranking matrix by a constant component. So the group decision determined with the use of methods based on an outranking matrix, e.g. the Condorcet or the Borda method, are – in both cases – the same.

It is worth noting that it is not possible to uniquely determine whether there were tied alternatives in experts' opinions or not from either the majority margin matrix or the outranking matrix. If a method for determining a group decision utilizes information about ties between alternatives, then the conclusion about the consistency of opinions generated might not be true.

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