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Example of spatial frequency shifting effect in Leith – Upatnieks holography

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An example of spatial frequency shifting in the real image of Leith – Upatnieks holography is shown. The due analysis points out that the shifting effect is responsible for the deformation of the real image. This is shown in the case of discrete angular spectrum and continuous angular spectrum of the object, but reduced to the situation when the reference and reconstructing waves are identical, both being plane.

1. Introduction

The textbooks on holography [1]–[8] give various explanations of the deformation suffered by the so-called real image obtained in the Leith – Upatnieks geometry. Below, you will find a "natural" explanation of the deforming effect based on spatial frequency treatment of the problem involving the concept of angular spectrum. For the sake of simplicity, the considerations will be reduced to the case when the reconstructing and the reference waves are identical to each other. Then, the structure of the optical field (wave) emerging from the hologram is described by the well known formula

$$U(x,y) = rt(x,y) = k(r|r|^2+r|u|^2+|r|^2u+r^2u^* \tag{1}$$

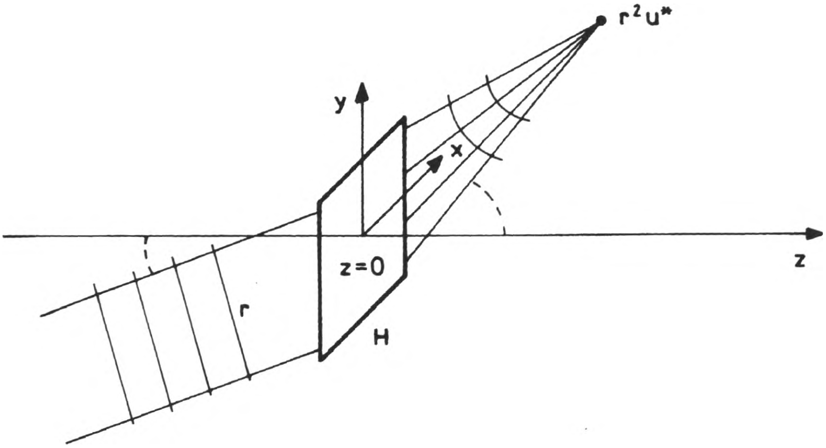
where k is a constant, r denotes the reference (and equivalently reconstructing) wave, u is the virtual image wave (identical with the object wave incident on the hologram), and u^* represents the so-called real image wave* being the subject of our considerations.

One can readily notice that the information about the object, *i.e.*, the object wave u, is encoded here in two terms $|r|^2u$ and r^2u^*, neither of which being strictly identical with u. However, it is equally easy to realize that the deformation suffered by the object wave u because of being represented by the term $|r|^2u$ is relatively small and concerns only its amplitude structure (as a result of $|r|^2$-factor operation), while its phase structure remains unchanged.

In contrast to this, the corresponding deformation of the real image u^* represented by the term r^2u^* is of both amplitude and phase nature. In order to

* Called also secondary image wave.

distinguish the perfect virtual u and real u^* images from their respective representative waves $|r|^2u$ and r^2u^* emerging in reality from the hologram during reconstruction, we shall call the first two waves the **pure virtual and pure real image waves**, while the other two — **the deformed virtual and deformed real image waves**, respectively. Our goal is to find the relation between the pure and deformed real images in holography of Leith—Upatnieks geometry (Figure).



Reconstruction of the real image in the simple Leith—Upatnieks geometry when the reference and reconstructing waves are identical and reduced to plane waves. H — hologram in the $z = 0$ position, r — reconstructing (= reference) plane wave, r^2u^* — deformed real image wave

For this purpose let us note that if

$$r = r_0(x, y, z)\exp[i\Phi_r(x, y, z)], \tag{2}$$

and

$$u = u_0(x, y, z)\exp[i\Phi_u(x, y, z)]. \tag{3}$$

Then the pure real image wave is

$$u^* = u_0(x, y, z)\exp[-i\Phi_u(x, y, z)], \tag{4}$$

and the deformed real image takes the form

$$u' = r_0^2(x, y, z)u_0(x, y, z)\exp[-i(\Phi_u(x, y, z) - 2\Phi_r(x, y, z)] \tag{5}$$

where $r_0(x, y, z)$ and $u_0(x, y, z)$ are the real amplitudes of the respective waves.

2. Case of discrete angular spectrum

In order to simplify the considerations, assume that at the hologram plane $z = 0$, the reconstructing wave r is a plane wave of the form

$$r = r_0\exp 2\pi i(f_{xr}x + f_{yr}y) = r_0\exp\frac{2\pi}{\lambda}(z\cos\alpha_r + y\cos\beta_r). \tag{6}$$

2.1. Regular discrete angular spectrum

Let the pure real image plane be representable in the form of Fourier series

$$u^* = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{mn}\exp[2\pi i(f_{xm}x+f_{yn}y)]$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-n}^{\infty} a_{mn}\exp\left[\frac{2\pi}{\lambda}(x\cos\alpha_m+y\cos\beta_n)\right] \tag{7}$$

where a_{mn} are the real amplitudes of the respective component plane waves (spatial harmonics), characterized either by the spatial frequencies

$$f_{xm} = mf_{x1} \quad \text{and} \quad f_{ym} = nf_{y2} \tag{8}$$

where f_{x1}, f_{y1} are the fundamental spatial frequencies and $n, m = 0, \pm 1, \pm 2, \ldots$, or equivalently by the directional cosines: $\cos\alpha_m$, $\cos\beta_n$ of the same component plane waves fulfilling the relations:

$$\cos\alpha_m = m\cos\alpha_1, \quad \cos\beta_n = \mathrm{n}\cos\beta_1 \tag{9}$$

where: $\cos\alpha_1 = \lambda f_{x1}$, $\cos\beta_1 = \lambda f_{y1}$.

Note that in accordance with (7a,b), the pure real image wave is assumed to be of **discrete angular spectrum of regularity** defined by (8) and (9), respectively.

2.2. Irregular discrete angular spectrum

In contrast to (6a), the deformed real image wave u' takes, in the face of (5) and (6), the form

$$u' = r^2 u^*$$

$$= |r|^2 \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{mn}\exp\left\{\frac{2\pi i}{\lambda}[x(\cos\alpha_m+2\cos\alpha_r)+y(\cos\beta_m+2\cos\beta_r)]\right\} \tag{10a}$$

$$= |r|^2 \sum_{m=-\infty}^{\infty}\sum_{n=-1}^{\infty} a_{mn}\exp\frac{2\pi}{\lambda}[k\cos\alpha'_m+y\cos\beta'_n] \tag{10b}$$

$$= |r|^2 \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{mn}\exp\{2\pi i[(f_{xm}+2f_{xr})r+f_{ym}+f_{yr})r]\} \tag{10c}$$

$$= |r|^2 \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{mn}\exp[2\pi i(f'_{xm}x+f'_{yn}y)]. \tag{10d}$$

Here, the following relations are valid:

$$f'_{xm} = f_{xm}+2f_{xr}, \quad f'_{ya} = f_y+2f_{yr}, \tag{11a}$$

$$f'_{xm} = \frac{\cos\alpha_m+2\cos\alpha_r}{\lambda} = \frac{\cos\alpha'_m}{\lambda}, \quad f'_{ym} = \frac{\cos\beta_m+2\cos\beta_r}{\lambda} = \frac{\cos\beta'_m}{\lambda}, \tag{11b}$$

$$\cos\alpha'_n = \cos\alpha_n+2\cos\alpha_r, \quad \cos\beta'_n = \cos\beta_n+2\cos\beta_r, \tag{11c}$$

where

$$f_{xr} = \frac{\cos\alpha_r}{\lambda}, \quad f_{yr} = \frac{\cos\beta_r}{\lambda} \quad \text{(cf. (5))}.$$

The set of formulae (11) indicates that the spatial frequencies of the deformed real image wave are each shifted by the same constant component (f_{xr}, f_{yr}) value with respect to the spatial frequencies of the pure real image wave. This means that the directions of propagation of the component plane waves constituting the deformed real image wave (10) are different from those of the pure real image wave (7), as indicated by relation (11b). Obviously, the same is true for the spatial frequency spectra in both cases as can be clearly seen when comparing formulae (11a) and (8). This constitutes the **irregularity** of the spatial frequency spectrum. This problem will be discussed later in a more complete way. Now, we intend to specify some other aspect of the pure real image—deformed real image relation.

2.3. Further discussion of the mutual relation of the pure and deformed real image waves

Relations (11a) and (11c) compared to the relation (8) indicate that the whole angular spectrum of the pure real image wave is changed, provided f_{xr} and f_{yr} are both different from zero. Now, we want to show that the angular spectrum of the deformed real image wave is essentially **irregular**, which means that it is impossible to present it in the regular form analogical to (8) or (9). For this purpose, note that

$$f'_{xm} = f_{xm} + 2f_{xr} = mf_{x1} + 2f_{xr} = m(f'_{x1} - 2f_{xr}) + 2f_{xr} = mf'_{x1} - 2(m-1)f_{xr} \neq mf'_{x1} \tag{12a}$$

and

$$f'_{yn} \neq nf'_{y1}. \tag{12b}$$

Here, we have taken advantage of the relations:

$$f_{x1} = f'_{x1} - 2f_{xr}, \; f_{y1} = f'_{y1} - 2f_{xr}$$

following from (11a).

Obviously, the same can be proved for the directional cosines which means that

$$\cos\alpha'_m \neq m\cos\alpha'_1 \quad \text{and} \quad \cos\beta'_n \neq n\cos\alpha'_1. \tag{12c}$$

This shows that the reconstruction of the real holographic image with a plane reconstruction wave r of the form (6) must result in deformation due to loss of the angular spectrum regularity (8) or (9). The regularity is preserved only if $f_{xr} = f_{yr} = 0$ (or equivalently if $\cos\alpha_r = \cos\beta_r = 0$).*

2.4. Concluding remarks

1. Due to the interaction of the reconstruction wave r of type (6) with each component plane wave $a_{mn}\exp[2\pi i(xf_{xm} + yf_{yn})]$ of the pure real image wave, the

* This means that the reconstructing (= reference) wave falls perpendicularly on the hologram plane.

latter is transformed into another plane wave $|r|^2 a_{mn} \exp[2\pi i(xf'_{xn}+yf'_{yn})]$ of other spatial frequency defined in (11a) and consequently propagating in another direction determined by new directional cosines defined in (11c).

2. Thus, the whole discrete angular spectrum of the pure real image wave is changed loosing its regularity (see (12b)) which is equivalent to essential changes in mutual directional structure of the original component plane waves of the pure real image wave; the direction of each component wave being changed to different degree.

3. This makes the application of the conventional transfer function to describe the real image deformation impossible and the latter should be replaced by a frequency shifting function.

3. Case of continuous angular spectrum

In a more general case when the object wave u is of continuous angular spectrum its pure real image wave u^* at the hologram plane $z = 0$ may be put in the form of Fourier integral

$$u^* = \iint_{-\infty}^{\infty} a(f_x,f_y)\exp 2\pi i(f_x x+f_y y)df_x df_y \tag{13a}$$

$$= \iint_{-\infty}^{\infty} a\left(\frac{\cos x}{\lambda}, \frac{\cos\beta}{\lambda}\right)\exp\frac{2\pi i}{\lambda}(x\cos\alpha+y\cos\beta)d\left(\frac{\cos\alpha}{\lambda}\right)d\left(\frac{\cos\beta}{\lambda}\right). \tag{13b}$$

In this case, the deformed real image wave u' takes the form

$$u' = ru^* = |r|^2 \iint_{-\infty}^{\infty} a(f_x,f_y)\exp 2\pi i[x(f_x+2f_{xr})+y(f_y+2f_{yr})]df_x df_y \tag{14a}$$

$$= \iint_{-\infty}^{\infty} a\left(\frac{\cos\alpha}{\lambda}, \frac{\cos\beta}{\lambda}\right)\exp\frac{2\pi i}{\lambda}[x(\cos\alpha+2\cos\alpha_r)+y(\cos\beta+2\cos\beta_r)]$$

$$\times d\left(\frac{\cos\alpha}{\lambda}\right)d\left(\frac{\cos\beta}{\lambda}\right). \tag{14b}$$

When comparing (13) and (14), it is visible that the shifting effect of the angular spectrum occurs also in the case of continuous angular spectrum. Keeping in mind the interpretation of the shifting consequences for the discrete case considered in Sect. 2, it is easy to imagine that also in continuous case the directional structure of the object angular spectrum is damaged in an analogous way. This means also that the original spectrum of the object (or pure real image) expressed in terms of spatial frequencies (f_{xm},f_{yn}), (or equivalently by directional cosines $(\cos\alpha_m, \cos\beta_n)$) is replaced by that expressed in frequencies $(f'_{xm} = f_{nm}+2f_{xr}, f'_{ym} = f_{ym}+2f_{yr})$ or equivalently by directional cosines $(\cos\alpha'_m = \cos\alpha_m+2\cos\alpha_r,\ \cos\beta'_n = \cos\beta_n+2\cos\beta_r)$.

A new feature of the case of continuous angular spectrum compared to that of the discrete angular spectrum is that now the shifting effect does not exclude, in principle, the application of the conventional optical transfer function, though the results of its application should be interpreted with greater care taking account of the shifting effect.

4. Final conclusions

1. Above, it has been shown that frequency shifting effect appears both in the discrete angular spectrum and continuous angular spectrum cases.

2. The shifting effect can be totally removed either by assuming both the reference and reconstructing waves incident perpendicularly to the hologram plane, *i.e.*, when $\cos\alpha_r = \cos\beta_r = 0$, or when one of them is complex conjugate of the other.

3. The application of the concept of the optical transfer function is impossible in the discrete angular spectrum case due to irregularity of this spectrum introduced by the reconstruction process (shifting effect), while the application of this function to the continuous angular spectrum case provides some interpretative difficulties.

4. The considerations carried out in this paper are idealized also due to the fact that some important perturbing effects like that of the diffraction at the hologram rim are still neglected.

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