

Poincaré sphere in the optics of anisotropic media

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In the paper, the various ways of application of the Poincaré sphere to the polarization optics and the optics of anisotropic media proposed in the literature have been reviewed and enriched. The considerations do not include ellipsometry being reduced to the most elementary case of transmitting and reflection of polarized light at the border of the isotropic media.

1. Introduction

The main advantage offered by Poincaré sphere to the polarization optics and the optics of anisotropic media is the possibility of obtaining quick approximate results useful for preliminary calculations when designing the polarization devices and for the course changes of the polarization state of the light in optical polarization systems. These operations are usually made in mind. Sometimes some easy auxiliary calculations appear necessary which do not exceed the possibilities of a simple calculator equipped with trigonometric functions.

2. Poincaré sphere

The vectors of the electric field strength \mathbf{E} of a polarized plane electromagnetic wave are distributed along the normal to the wavefront so that their ends create a helix of equation

$$E_x = m_x \cos(\omega t - kz - \delta_x),$$

$$E_y = m_y \cos(\omega t - kz - \delta_y),$$

where: E_x, E_y – Cartesian components of the vector \mathbf{E} , m_x, m_y – real amplitudes of the component vectors $\mathbf{E}_x, \mathbf{E}_y$, $\omega = \frac{2\pi}{T}$ (T – period), t – time, $k = \frac{2\pi}{\lambda}$ (λ – wavelength), z – distance in the direction normal to the wavefront, δ_x, δ_y – initial phases (for $t = 0$, and $z = 0$) of the component $\mathbf{E}_x, \mathbf{E}_y$.

This helix passing through the plane $z = \text{const}$ creates a trace of equations:

$$E_x = m_x \cos \omega t,$$

$$E_y = m_y \cos(\omega t + \delta) \tag{1}$$

where $\delta = \delta_x - \delta_y$, in the latter called the polarization state ellipse, shown in Fig. 1. Besides the physical quantities already defined some other are visible, namely:

– azimuth α – the angle subtended by the great axis of the polarization state ellipse with the x -axis of the coordinate system,

– β – diagonal angle defined as $\beta = \arctan \frac{m_y}{m_x}$,

– ϑ – ellipticity angle defined as $\vartheta = \arctan \frac{b}{a}$ (a, b – great and small respective half-axes of the ellipse).

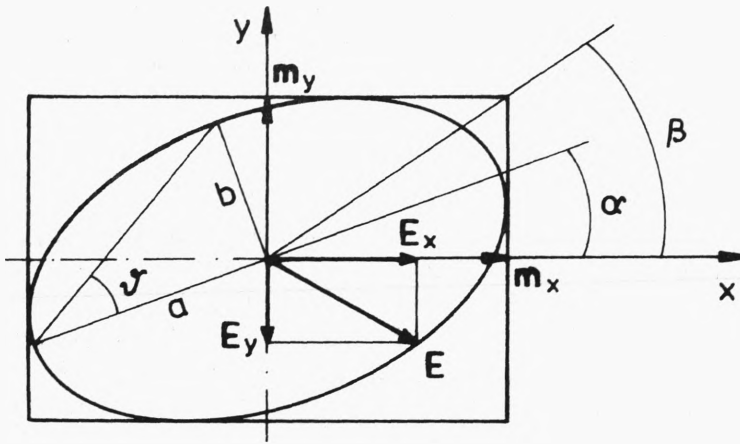


Fig. 1. Ellipse of the light polarization state

The intensity and the light polarization state can be, among others, described by the Stokes vector and the Poincaré sphere. The Stokes vector appears in two variants: as functions of α, ϑ , i.e., $[S(\alpha, \vartheta)]$ and β, δ , i.e., $[S(\beta, \delta)]$. This gives

$$[S(\alpha, \vartheta)] = \begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} = \begin{bmatrix} I \\ pIM_u \\ pIC_u \\ pIS_u \end{bmatrix} = \begin{bmatrix} I \\ pI \cos 2\vartheta \cos 2\alpha \\ pI \cos 2\vartheta \sin 2\alpha \\ pI \sin 2\vartheta \end{bmatrix}, \quad (2)$$

$$[S(\beta, \delta)] = \begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} = \begin{bmatrix} I \\ pIM_u \\ pIC_u \\ pIS_u \end{bmatrix} = \begin{bmatrix} I \\ pI \cos 2\beta \\ pI \sin 2\beta \cos \delta \\ pI \sin 2\beta \sin \delta \end{bmatrix} \quad (3)$$

where I denotes the light intensity, p – degree of polarization, M, C, S – the Cartesian coordinates of the end of the vector pI , the direction of which is determined by the spherical coordinates $2\alpha, 2\vartheta$ (Fig. 2), or the coordinates $2\beta, \delta$ (Fig. 3). In Figure 3, the vector pI creates the angle 2β with the axis M and δ is the

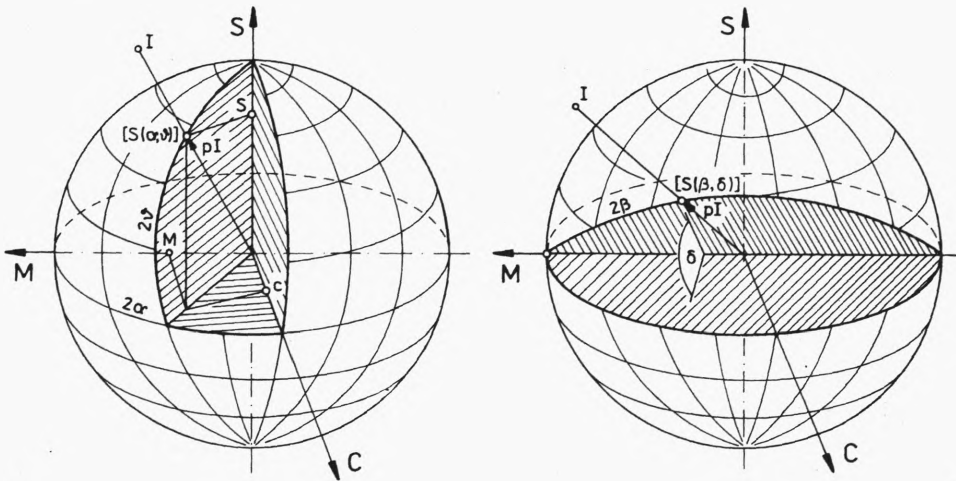


Fig. 3

Fig. 2. Assignment of the point $(2\alpha, 2\vartheta)$ to the Stokes vector $[S(\alpha, \vartheta)]$ of the polarization state and to the associated vector pI of the intensity of the polarized part of the light. The scalar value of the intensity of the unpolarized part of the light is put on the prolongation of the vector pI

Fig. 3. Assignment of the point $(2\beta, \delta)$ to the Stokes vector $[S(\beta, \delta)]$ of the light polarization and to the connected vector pI of the intensity of the polarized light. The scalar value of the intensity of the unpolarized light is put on the prolongation of the vector pI

angle between the equator plane containing the axes M and C and the plane determined by the axis M and the vector pI . M_u, C_u, S_u are the Cartesian coordinates of the unity Poincaré sphere, *i.e.*, that of unity radius. The name Poincaré sphere is of historical character. The coordinates M, C, S describe the sphere of radius pI equal to the intensity of the polarized part of the light beam. When several light beams are superposed or if a light beam passes through any medium or is subject to reflection pI is not constant. Therefore the sphere radius should be assumed arbitrary, for instance, equal to unity and the values of I and pI can be laid along the direction $2\alpha, 2\vartheta$ (or equivalently $2\beta, \delta$). This direction hits the sphere surface at a point to which the polarization state α, ϑ (or equivalently β, δ) should be prescribed. In addition to the intensity and the polarization state of the light also the Stokes vectors of the eigenwaves of the birefringent media and polarizers may be localized on the Poincaré sphere, the values $I = 1$ and $p = 1$ being arbitrarily prescribed to them. This is an additional argument for assuming the standard radius of the sphere equal to unity. It should be added that the Stokes vector of the polarization state of the light is a peculiar vector. Its components M, C, S added vectorially to each other are equal to the intensity pI of the polarized part of the light beam, which is a scalar quantity. The vectorial character has its state of polarization. The scalar product of the unitary components M_{u1}, C_{u1}, S_{u1} and M_{u2}, C_{u2}, S_{u2} of two different polarization states $[S_1]$ and $[S_2]$ is equal to the cosine of the angle c between the vectors. The first "component" I of the Stokes vector is the sum of the intensities of the polarized and

unpolarized parts of the light and is not a vector. It cannot be vectorially added neither multiplied scalarly* or vectorially with components M , C , S . It is only for the sake of convenience that it will be presented here graphically as a scalar overlapping with the vector pI and having a common origine with the latter in the centre of the Poincaré sphere.

On the equator of the sphere the linear polarization states of different azimuths are located while at the poles — the circular polarization states. The more general elliptic polarization states of different ellipticity ϑ and equal azimuths α are located on the meridians while those of equal ellipticities and different azimuths — on the parallels. The dextrorotatory states of polarization occupy the northern hemisphere while the laevorotatory ones are placed on the southern hemisphere.

3. Generalized Malus law

A partially polarized light beam of intensity I_0 and the degree of polarization p_0 and defined by the Stokes vector $[S_0] = [I_0, M_0, C_0, S_0]$, (Fig. 4), passes through a birefringent medium. The components of the first Stokes eigenvector $[S]$ of the latter are: 1, M , C , S and the energetic transmission of the faster and slower eigenwaves are t_1 and t_2 , respectively.

The light intensity I after passing through the medium is determined by the Malus law [1]

$$I = 0.5I_0(t^+ + p_0t^- \cos c) \quad (4)$$

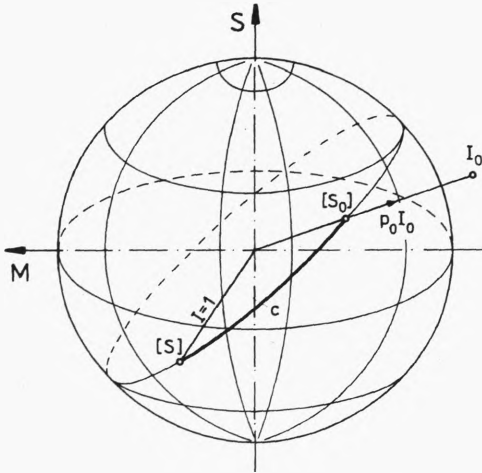


Fig. 4. Light of intensity I_0 and the polarization degrees p_0 defined by Stokes vector $[S_0]$ passes through the dichroic medium of Stokes vector $[S]$, c — central angle between the vectors $[S]$ and $[S_0]$

* In the sense of scalar product of vectors.

where: $t^+ = t_1 + t_2$; $t^- = t_1 - t_2$;

t_1 — power transmission coefficient of the first (faster) eigenwave,

t_2 — power transmission coefficient of the second (slower) eigenwave,

$\cos c = (MM_{u0} + CC_{u0} + SS_{u0})$,

c — angular distance of the vectors $[S]$ and $[S_0]$ joined by an arc of the great circle on the Poincaré sphere.

The degree of polarization of the light emerging from the medium is

$$p = \frac{(1 - p_0)t^- + p_0(t^+ + t^- \cos c)}{t^+ + p_0 t^- \cos c} \tag{5}$$

The intensity of the light passing through an arbitrary dichroic birefringent medium $[S]$ is the same for all the polarization states $[S_0]$ equally distant from the point $[S]$.

4. Intensities of the eigenwaves in a birefringent medium

A partially polarized light wave defined by the Stokes vector $[S_0]$ is incident on a birefringent medium (or a polarizer), the Stokes vector of the first (faster) eigenwave of which is denoted by $[S_1]$, (Fig. 5), and that of the other (slower) by $[S_2]$.

The vectors $[S_0]$, $[S_1]$ and $[S_2]$ lie on the Poincaré sphere on the common great circle. The angular distance between $[S_0]$ and $[S_1]$ amounts to c .

The intensities of both eigenwaves diminish in the medium. At the beginning of the medium they amount to [2]:

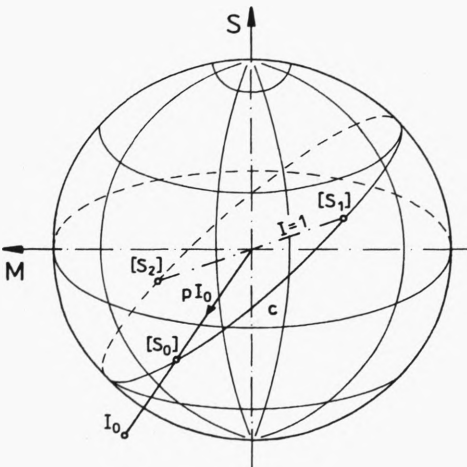


Fig. 5. Partially polarized light determined by the Stokes vector $[S_0]$ falls onto a birefringent dichroic medium the eigenwaves of which are described by the vectors $[S_1]$ and $[S_2]$, c — central angle between the vector $[S_0]$ and the first eigenvector of the medium $[S_1]$

$$\begin{aligned} I_1 &= 0.5I_0(1 + p_0 \cos c), \\ I_2 &= 0.5I_0(1 - p_0 \cos c), \end{aligned} \quad (6)$$

while at the end respectively

$$\begin{aligned} I_1 &= 0.5t_1 I_0(1 + p_0 \cos c), \\ I_2 &= 0.5t_2 I_0(1 - p_0 \cos c). \end{aligned} \quad (7)$$

5. Change of the light polarization state while passing through a birefringent medium

Let a medium be given the eigenwaves of which are described by the Stokes vectors $[S_1]$ and $[S_2]$, (Fig. 6). A partially polarized light beam described by the Stokes vector $[S_0]$ hits the medium. Our goal is to determine the state of polarization $[S]$, intensity and the degree of polarization of the light after leaving the medium.

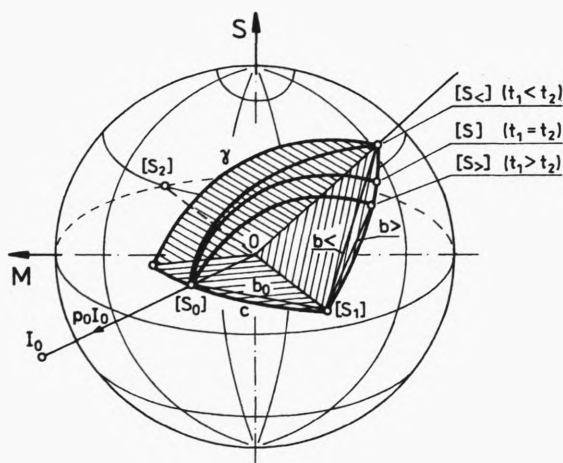


Fig. 6. Partially polarized light described by the Stokes vector $[S_0]$ falls onto the dichroic medium the first eigenwave of which is described by the vector $[S_1]$ distant by an angle c from $[S_0]$. After having passed through the medium which introduced the phase shift γ between the eigenwaves the polarization state $[S]$, $[S_<]$, or $[S_>]$ of the light emerging lies on the arc of the great circle creating the angle γ with the arc $[S_0]$, $[S_1]$. When the medium is not dichroic the distance of the vector $[S]$ from $[S_1]$ of the emerging light is the same as that of the incident light. In the dichroic media it is less (for $t_1 > t_2$) or greater ($t_1 < t_2$); b_0, b_1, b_2, b_3, b_4 — chords

The procedure is as follows.

1. The first eigenvector $[S_1]$ of the medium and the vector $[S_0]$ of the polarization state of the incident light should be joined by the due arc of the great circle.
2. The arc of the great circle rotates on the Poincaré sphere around the point $[S_1]$ by the angle γ of the mutual shift of phases of two eigenwaves of the birefringent medium.

3. The vector of the end polarization state $[S]$ lies on the rotated arc and creates with the point $[S_1]$ a chord of length $[3]$

$$b = b_0 \sqrt{\frac{2t_2}{t^+ + t^- \cos c}} \tag{8}$$

where: b_0 is the length of the chord between the initial points $[S_1]$ and $[S_0]$, and c is the central angle between those points:

$$t^+ \equiv t_1 + t_2; \quad t^- \equiv t_1 - t_2.$$

Practical indication

In the case of nondichroic medium, (Fig. 7), ($t_1 = t_2$, $b = b_0$) the point of the compasses can be located in the point $[S_1]$, the pencil on the point $[S_0]$ and an arc of angle γ should be drawn clockwise.

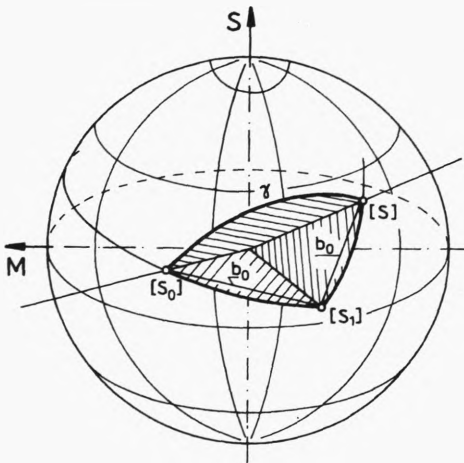


Fig. 7. The same as in Fig. 6, but for the nondichroic medium

In the case of dichroic medium the compasses opening should be set to the value b (see Eq. (8)) and an arc of angle γ should be drawn starting with the beginning of the arc passing through $[S_1]$ and $[S_0]$. The emerging light intensity should be calculated from the generalized Malus law (4) while its degree of polarization – from the formula (5).

6. Measurement principle for the light polarization state

The state of polarization $[S_0] = f(\alpha_0, \vartheta_0, p_0, I_0)$ of the light beam must be examined. For this purpose [4], this beam is transmitted successively through the three real polarizers $[S_1]$, $[S_2]$ and $[S_3]$, (Fig. 8), measuring each time the light intensity I_i

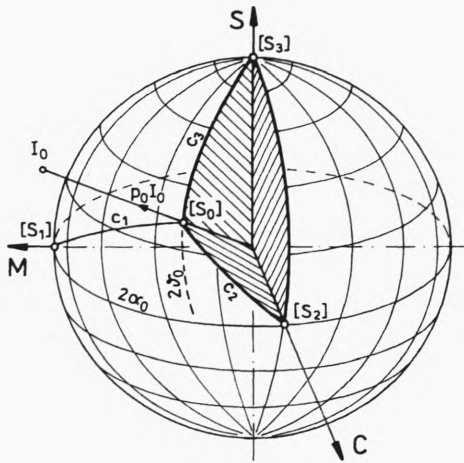


Fig. 8. Light of unknown both the state of polarization $[S_0]$ and the degree of polarization passes through the polarizer $[S_1]$, then through $[S_2]$ and finally through $[S_3]$. The intensity of light transmitted through these polarizers allow to calculate the state and degree of polarization as well as the intensity I_0 of the light examined

of the transmitted light ($i = 1, 2, 3$). From the generalized Malus law [1]

$$2I_i = I_{0i}t_i^+ + t_i^-(M_0M_i + C_0C_i + S_0S_i).$$

In other words, (see formula (4))

$$2I_i = I_{0i}(t_i^+ + p_0 t_i^- \cos c_i).$$

where: c_i – angle between $[S_0]$ and $[S_i]$,

$$\cos c_i = M_{u0}M_i + C_{u0}C_i + S_{u0}S_i.$$

After some substitutions

$$\frac{2 \frac{I_i}{I_{0i}} - t_i^+}{t_i^-} = A_i$$

where: $t^+ = t_1 + t_2$, $t^- = t_1 - t_2$.

For the polarizers: a linear one of azimuth $\alpha_1 = 0$ ($[S_1]$), a linear one of azimuth $\alpha_2 = 45^\circ$ ($[S_2]$) and circular dextrorotatory ($[S_3]$) the equality (4) after some manipulations takes the form

$$\begin{aligned} A_1 &= p_0 \cos 2\vartheta_0 \cos 2\alpha_0, \\ A_2 &= p_0 \cos 2\vartheta_0 \sin 2\alpha_0, \\ A_3 &= p_0 \sin 2\vartheta_0. \end{aligned} \tag{9}$$

The magnitudes determining the state of polarization of the light under study are finally calculated from the relations:

$$\begin{aligned}
 p_0 &= \sqrt{A_1^2 + A_2^2 + A_3^2}, \\
 \vartheta_0 &= 0.5 \arcsin\left(\frac{A_3}{p_0}\right), \\
 \alpha_0 &= 0.5 \arccos\left(\frac{A_1}{p_0 \cos 2\vartheta_0}\right).
 \end{aligned}
 \tag{10}$$

For: $A_2 > 0, \quad 0^\circ < \alpha_0 < 90^\circ,$
 $A_2 < 0, \quad -90^\circ < \alpha_0 < 0^\circ.$

7. Polarizing features of a nondichroic medium

There exist several methods of measurement of the birefringence medium polarizing features, the basis of which being the measurement of the change of polarization state due to passage of light through this medium. In the most frequent case when the medium is not dichroic an approximate result may be found graphically on the Poincaré sphere (Fig. 9), while the exact one can be calculated from relations presented in [5].

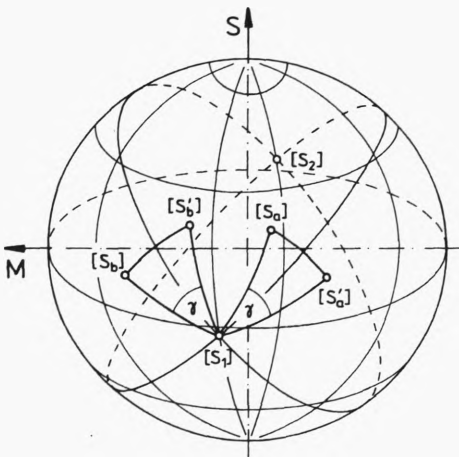


Fig. 9. Light $[S_a]$ after passing through a nondichroic medium $[S_1]$ has the polarization state $[S'_a]$. The light of different state of polarization $[S_b]$ after passing through the same medium has the polarization state $[S'_b]$. The two points of intersection of the great circles of which one passes perpendicularly through the middle of the arc $[S_a], [S'_a]$ and the other being that of $[S_b], [S'_b]$, determine the Stokes vectors $[S_1]$ and $[S_2]$ of the eigenwaves of the medium. The angle γ in both cases is the same

In order to find the states of polarization $[S_1]$ and $[S_2]$ of the two eigenwaves of the medium the latter must be illuminated with the light of polarization state $[S_a]$ which after passing through this medium has the polarization state $[S'_a]$. Next, the same medium should be illuminated with the light of polarization state $[S_b]$, which

after passing through the medium gets the polarization state $[S'_b]$. In the Poincaré sphere two great circles are to be drawn. One perpendicularly to the arc $[S_a], [S'_a]$ and passing through its centre and the other perpendicular to the arc $[S_b], [S'_b]$. Both intersection points of the great circles determine the vectors $[S_1]$ and $[S_2]$ of the two eigenwaves of the medium. Which of these two intersection points corresponds to the first eigenwave, should be determined by an additional measurement (made, for instance, with a polariscope equipped with a phase plate).

The calculations based on the above reasoning lead to the relationships determining the state of polarization of the eigenwaves of the medium [5].

The components M_1, C_1, S_1 of the sought eigenvector $[S_1]$ of the examined medium are:

$$M_1 = \pm \left\{ \left[1 + \left(\frac{M_{ua} - M_{ua'}}{S_{ua} - S_{ua'}} \right)^2 \right] - 2 \frac{L_M (M_{ua} - M_{ua'}) (C_{ua} - C_{ua'})}{L_C (S_{ua} - S_{ua'})^2} + \left(\frac{L_M}{L_C} \right)^2 \left[1 + \left(\frac{C_{ua} - C_{ua'}}{S_{ua} - S_{ua'}} \right)^2 \right] \right\}^{-1/2},$$

$$C_1 = -M_1 L_M / L_C,$$

$$S_1 = -[M_1 (M_{ua} - M_{ua'}) + C_1 (C_{ua} - C_{ua'})] / (S_{ua} - S_{ua'}),$$

$$\gamma = \arccos[(\cos c_{aa'} - \cos^2 c_{1a}) / \sin^2 c_{1a}],$$

where:

$$L_M = (M_{ub} - M_{ub'}) - (M_{ua} - M_{ua'}) (S_{ub} - S_{ub'}) / (S_{ua} - S_{ub}),$$

$$L_C = (C_{ub} - C_{ub'}) - (C_{ua} - C_{ua'}) (S_{ub} - S_{ub'}) / (S_{ua} - S_{ub}),$$

$$\cos c_{aa'} = M_{ua} M_{ua'} + C_{ua} C_{ua'} + S_{ua} S_{ua'}, \quad C_{aa'}, \quad \cos c_{1a} = M_1 M_{ua} + C_1 C_{ua} + S_1 S_{ua}.$$

From the first eigenvector calculated in this way the properties of the medium can be determined:

$$\vartheta_1 = 0.5 \arcsin S_1, \quad \alpha_1 = \arccos(M_1 / \cos 2\vartheta_1).$$

If $C_1 > 0$, then $0^\circ < \alpha_1 < 90^\circ$, if $C_1 < 0$, then $0^\circ > \alpha_1 > -90^\circ$

where the indices a and b refer to the polarization states of the testing beams while a' and b' – to the corresponding states of polarization of the transmitted beams.

The second set of results corresponding to the point $[S_2]$ on the sphere is

$$\alpha_2 = \alpha_1 + 90^\circ, \quad \vartheta_2 = -\vartheta_1, \quad \gamma_2 = 360^\circ - \gamma_1.$$

As already mentioned without additional measurement it cannot be decided which set of results refers to the first (*i.e.*, faster) eigenwave.

8. Phase superposition principle for birefringent medium

Let an elliptically birefringent dichroic medium $[S(\alpha, \vartheta)]$ be given (Fig. 10) of the phase difference γ of the eigenwaves. A linearly polarized light beam described by

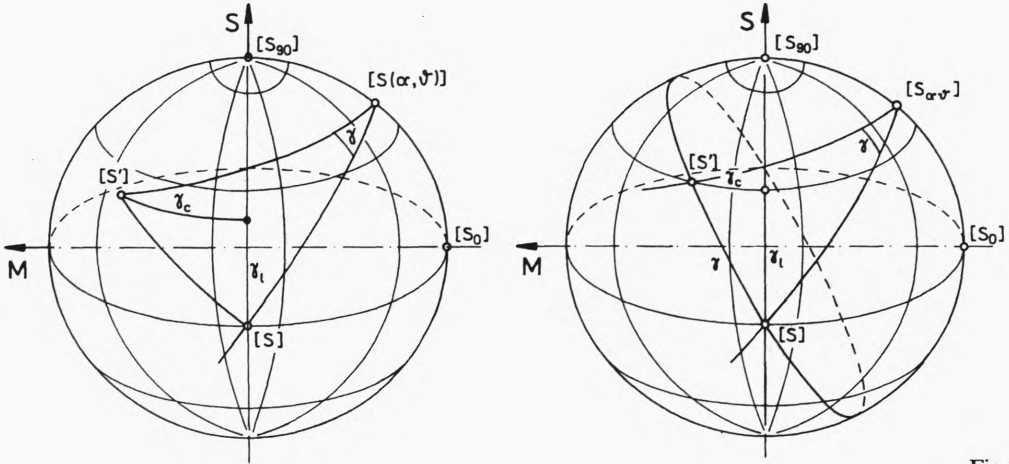


Fig.11

Fig. 10. Linearly polarized light $[S]$ passes through an elliptically birefringent medium $[S(\alpha, \vartheta)]$ of azimuth greater by 45° . The polarization state $[S']$ of the light transmitted can be obtained by using a replacement polarizing system composed of a linear phase plate $[S_0]$ of the same azimuth α as that of the elliptic medium and of a phase difference γ_t and a right-circular phase plate $[S_{90}]$ with a phase difference γ_c

Fig. 11. The same as in Fig. 10, but for nondichroic medium

Stokes vector $[S]$ of azimuth less by 45° than that of the medium $[S(\alpha, \vartheta)]$ falls onto this medium and consequently is transformed to the polarization state $[S']$. In order to obtain the same final polarization state $[S']$ for the linearly polarized incident wave $[S]$ as that produced by the medium, it can be replaced by two phase plates: a linear one $[S_0]$ of the same azimuth as that of $[S(\alpha, \vartheta)]$ and phase difference of phase γ_t and a right circular one $[S_{90}]$ of phase difference γ_c . Both phase differences are defined by the formulae [6]:

$$\begin{aligned} \sin \gamma_t &= \sin 2\vartheta \cos 2B + \cos 2\vartheta \sin 2B \sin \gamma, \\ \tan \gamma_c &= \sin 2\vartheta \tan \gamma - \frac{\cos 2\vartheta}{\tan 2B \cos \gamma}, \\ t_t t_c &= \frac{t_f + t_s}{2}, \\ \tan^2 B &= \frac{t_f}{t_s}. \end{aligned} \tag{11}$$

Besides, the relation called superposition principle is fulfilled

$$\cos \gamma_t \cos \gamma_c = \cos \gamma \sin 2B.$$

In a particular case when the ellipticity birefringent medium is not dichroic, $B = 45^\circ$, the above formulae simplify to the form [7]

$$\begin{aligned}
 \sin \gamma_i &= \cos 2\vartheta \sin \gamma, \\
 \tan \gamma_c &= \sin 2\vartheta \tan \gamma, \\
 t_i t_c &= t, \\
 \cos \gamma_i \cos \gamma_c &= \cos \gamma.
 \end{aligned}
 \tag{12}$$

This case is illustrated in Fig. 11. The linearly polarized light defined by the Stokes vector $[S]$ is incident on the medium of the first vector $[S(\alpha, \vartheta)]$ with the phase difference γ the azimuth of which is by 45° greater than that of the incident light. At the exit of the medium the light has the polarization state $[S']$. The same state of polarization $[S']$ is obtained when the light $[S]$ passes first through the linear plate $[S_0]$ with the phase difference γ_i and next through the right circular plate $[S_{90}]$ with the phase difference γ_c .

9. Polarization state of superposition of mutually incoherent light beams partially polarized

Two light beams of polarization states $[S_1]$ and $[S_2]$ are given, (Fig. 12). The total intensity of light of these beams are I_1 and I_2 , while the intensities of the polarized parts of these beams are $p_1 I_1$ and $p_2 I_2$, respectively. Both these beams superimpose.

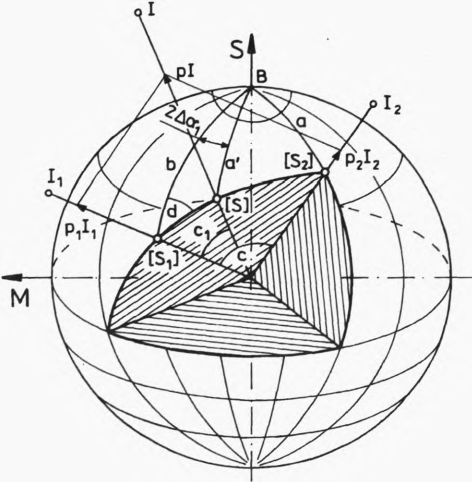


Fig. 12. When light-beam polarization state $[S_1]$ is superimposed on a light beam of polarization state $[S_2]$ and both the beams are mutually incoherent, the resultant state of polarization $[S]$ is determined by the intersection point of Poincaré sphere with the resultant intensity vector pI being the vectorial sum of the component intensities $p_1 I_1$ and $p_2 I_2$. The total intensity of light is equal to the scalar sum of intensities of both beams

What is the polarization state of the superposition? In the literature [8], the following answer is presented:

- The intensity of the polarized part of the resultant beam is a vectorial sum of

the polarized parts of the component beams. The intersection point of this vector and the Poincaré sphere determines the resultant polarization state of the superposed beams.

– The intensity I of the resultant beam is equal to the sum of intensities of the component beams

$$I = I_1 + I_2. \quad (13)$$

The degree of polarization of the resultant beam is

$$p = \frac{\sqrt{p_1^2 I_1^2 + p_2^2 I_2^2 + 2p_1 p_2 I_1 I_2 \cos c}}{I_1 + I_2}. \quad (14)$$

The angle made by the resultant vector $[S]$ and the component beam vector $[S_1]$ amounts to

$$c_1 = \arctan \left(\frac{p_2 I_2 \sin c}{p_1 I_1 + p_2 I_2 \cos c} \right). \quad (15)$$

– The ellipse (coordinates 2α , 2ϑ) of the resultant state of polarization $[S]$, in other words, the point of intersection of the Poincaré sphere by the resultant vector is determined graphically or analytically from the cosine law of spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos d \quad (16)$$

where: $b = 90 - 2\vartheta_1$,

$$a = 90 - 2\vartheta_2,$$

c – central angle of the arc $[S_1]$, $[S_2]$ of the great circle,

d – angle between the arc c and meridian b passing through the point $[S_1]$.

The formula (16) is applicable twice. First, from the triangle $[S_1]$, B , $[S_2]$ the angle d is calculated, next the value a' is calculated from the triangle $[S_1]$, B , $[S]$. The angle $2\Delta\alpha_1$ can be calculated from the sine law

$$\frac{\sin c_1}{\sin 2\Delta\alpha_1} = \frac{\sin a'}{\sin d}.$$

Finally, the new polarization state is determined by

$$2\alpha = 2\alpha_1 + 2\Delta\alpha_1,$$

$$2\vartheta = 90 - a'.$$

The above suggestions concerning the analytical calculations create a kind of completion of the literature data.

10. Polarization state of the superposition of mutually coherent polarized light beams

Two mutually coherent monochromatic light beams of intensities I_1 and I_2 and the Stokes vectors of the polarization light $[S_1]$ and $[S_2]$ lie on the Poincaré sphere in

mutual angular distance c [9]. The mutual phase shifts Δ of both beams are determined by the addition of Jones vectors of both waves

$$J = J_1 + J_2 e^{i\Delta}. \quad (17)$$

The relative intensity ratio of both beams is determined by the parameter

$$\mu = \arccos \sqrt{\frac{I_1}{I_1 + I_2}}. \quad (18)$$

The resultant polarization state is determined for two fundamental cases of interference:

1. When the ratio of the relative beam intensities is constant ($\mu = \text{const.}$), but their mutual phase shift Δ changes.

2. When the phase shift Δ is constant, but the parameter μ changes.

The polarization state $[S]$ of the light beam created due to superposition of two mutually coherent light beams $[S_1]$ and $[S_2]$ is expressed graphically in easiest way on an auxiliary primed Poincaré sphere M', C', S' superimposed on the primary one so that its equator passes through the points $[S_1]$ and $[S_2]$ and its zero meridian divides the arc $[S_1], [S_2]$ into two halves. The point $[S_2]$ should have a negative "geographic length", and $[S_1]$ – positive.

In the primed system, the Stokes vector $[S']$ of the polarization state occurring due to superposition of two mutually coherent light beams is calculated from the formula

$$[S'] = I \begin{bmatrix} 1 \\ \frac{\cos 2\alpha' + \sin 2\mu \cos \Delta}{1 + \cos 2\alpha' \sin 2\mu \cos \Delta} \\ \frac{\sin 2\alpha' \cos 2\mu}{1 + \cos 2\alpha' \sin 2\mu \cos \Delta} \\ \frac{\sin 2\alpha' \sin 2\mu \sin \Delta}{1 + \cos 2\alpha' \sin 2\mu \cos \Delta} \end{bmatrix}. \quad (19)$$

10.1. Case of $\mu = \text{const.}$, Δ – changing

The polarization states $[S']$ of the beam, being the result of interference of two coherent beams (Fig. 13) of polarization states $[S_1]$ and $[S_2]$ and the constant relative intensity defined by the parameter μ and of changing value of the phase shift Δ are positioned on the circle of the Poincaré sphere. In the primed sphere, the centre $O_\mu(2\alpha'_\mu, 2\vartheta'_\mu)$ of this circle lies on the great circle joining the vectors $[S_1]$ and $[S_2]$ of the interfering beams in the point of coordinates

$$2\alpha'_\mu = \arctan \frac{\tan \frac{c}{2}}{\cos \mu}, \quad 2\vartheta'_\mu = 0 \quad (20)$$

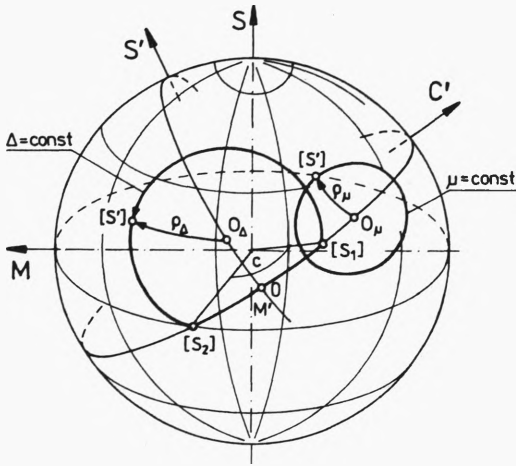


Fig. 13. Mutually coherent light beams of polarization states $[S_1]$ and $[S_2]$, distant from each other by angle c , interfere with one another. When the ratio of the light intensities of both beams is constant ($\mu = \text{const}$), the resultant states of polarization $[S']$ for different phase shifts Δ between the beams lie on circle of angular radius ρ_μ and the center O_μ . When the phase difference Δ is constant and the ratio of intensities of the interfering beams is changing, the Stokes vectors $[S']$ of the resultant polarization state lie on the arc spread between $[S_1]$ (beginning) and $[S_2]$ (end) of the angular radius ρ_Δ and the centre O_Δ

where: c – angular distance of the states $[S_1]$ and $[S_2]$ of the component beams,

$$\cos c = M_{u1} M_{u2} + C_{u1} C_{u2} + S_{u1} S_{u2}.$$

The central angle of its radius is

$$\rho_\mu = \arctan \left(\tan 2\mu \sin \frac{c}{2} \right). \tag{21}$$

The polarization states appearing for different phase shifts Δ are distributed along this circle both on the primed and nonprimed spheres.

10.2. Case of $\Delta = \text{const}$, μ – changing

The polarization states $[S']$ of the beam resulting from the interference of two coherent beams of polarization states $[S_1]$ and $[S_2]$ shifted in phase by Δ but having changeable values of the relative intensity (μ parameter) are positioned on the Poincaré sphere on the arc of the circle drawn in anticlockwise direction from $[S_1]$ to $[S_2]$. On the primed sphere the centre $O_\Delta(2\alpha_\Delta, 2\vartheta_\Delta)$ of this circle has the coordinates:

$$\begin{aligned} 2\alpha'_\Delta &= 0, \\ 2\vartheta'_\Delta &= c \tan \Delta \sin \frac{c}{2}. \end{aligned} \tag{22}$$

The central angle of its radius is

$$\rho_\Delta = \arctan \frac{\tan(c/2)}{\sin \Delta} \tag{23}$$

This arc determines the polarization states for different values μ of both component beams both on the primed and nonprimed spheres.

11. Change of the polarization state of the light at the border of dielectric isotropic media

The light falling at the border between the of isotropic dielectrics of refractive indices n_1 and n_2 is of polarization state determined by: the diagonal angle β_i (i – incident), the phase difference δ_i , the formula analogical to (1) and also Stokes matrix of type (3). In Figure 14, this is represented by the point $(2\beta_i, \delta_i)$, denoted as $[S_i]$. The real amplitudes m_{\parallel} and m_{\perp} of the components E_{\parallel} and E_{\perp} of the electric vector E of the incident wave (reflected and refracted waves) are respectively parallel (\parallel) and perpendicular (\perp) to the incidence plane; $\beta_{i,r,t} = \arctan(m_{\perp}/m_{\parallel})_{i,r,t}$.

11.1. Passing through the border of media

During passing through a border surface both amplitudes of the above mentioned two components change in accordance with the Fresnel formulae but no phase difference appears between them. The state of polarization is determined by:

$$\begin{aligned} \delta_t &= \delta_i, \\ \tan \beta_t &= \cos(\varphi - \psi) \tan \beta_i \end{aligned} \tag{24}$$

where: φ – incidence angle, ψ – refracting angle.

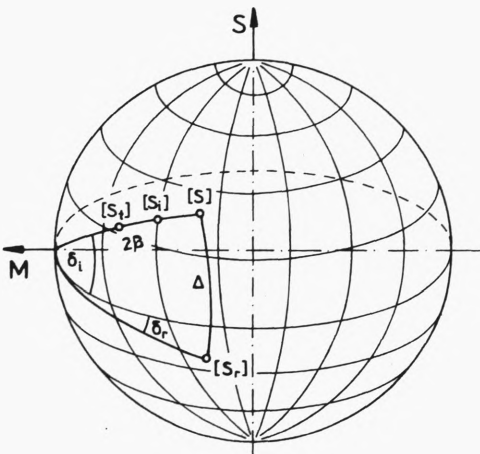


Fig. 14. Change of the polarization state $[S_i]$ of incident light after the refraction $[S_t]$ and reflection $[S_r]$ from the border of the media

$0 \leq \cos(\varphi - \psi) \leq 1$, thus the Stokes vector $[S_r]$, (Fig. 14,) lies on the same great circle of the Poincaré sphere as $[S_i]$, but closer than $[S_i]$ to the origin of the coordinate system.

11.2. Reflection of the light at the media border

The change of the diagonal angle is defined by the formula

$$\tan \beta_r = \left| \frac{\cos(\varphi - \psi)}{\cos(\varphi + \psi)} \right| \tan \beta_i, \quad (25)$$

and the angle $\delta_r = \delta_i - \Delta$ (Δ – phase jump during reflection). From the Fresnel formulae it follows that:

– The angle Δ amounts to π , for $n_1 < n_2$ and $0 < \varphi < \varphi_{br}$, and is equal to zero for $\varphi > \varphi_{br}$.

– The angle Δ amounts to π for $n_1 > n_2$ and $0 < \varphi < \varphi_{lim}$, and is equal to zero for $\varphi_{br} < \varphi < \varphi_{lim}$ and for $\varphi > \varphi_{lim}$.

$$\tan \frac{\Delta}{2} = \frac{\cos \varphi \sqrt{\sin^2 \varphi - n^2}}{\sin^2 \varphi},$$

where: φ_{br} – Brewster angle, φ_{lim} – limiting angle, $n = n_2/n_1$ (n_1 – refraction index of the medium of incident and reflected light, n_2 – refraction index of the medium of transmitted light).

For the metals, Δ is different from zero in the whole range of φ . In order to determine the state of polarization of the reflected wave (Fig. 14), first we denote the point $[S]$ on the arc $O[S_i]$, and next we turn it around the axis M by the angle Δ in the clockwise direction to the point $[S_r]$.

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