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Piotr Maćkowiak *

ADAPTIVE ROLLING PLANS ARE GOOD

Here we prove the goodness property of adaptive rolling plans in a multisector optimal growth model under decreasing returns in deterministic environment. Goodness is achieved as a result of fast convergence (at an asymptotically geometric rate) of the rolling plan to balanced growth path. Further on, while searching for goodness, we give a new proof of strong concavity of an indirect utility function – this result is achieved just with help of some elementary matrix algebra and differential calculus.

Keywords: indirect utility function, good plans, adaptive rolling planning, multisector model.

JEL classification: C61, O41

1. INTRODUCTION

The idea of this paper comes from Kaganovich (1996) where a hypothesis is put forward that adaptive rolling plans are good (in Gale's sense, see Gale (1967)), if there is uniform strong convexity of technology. Goodness of rolling plans in one sector models was proven in Bala et al. (1991). Fast convergence (at an asymptotically geometric rate) toward turnpike under linear technology (with suitably defined opitmality criterion) is known from Kaganovich (1998). We extend these results to the case where production of goods is described by neoclassical technology.

Rolling planning is a procedure of constructing infinite horizon programs. After finding an optimal process starting from a given initial state and under a fixed and finite horizon length of planning, only the first step of the plan is executed and a new optimal plan is constructed starting from the just achieved state (Goldman 1968). When feasible processes of growth are those in which initial and final state of the economy is the same (changes may occur between initial and final periods), then we deal with adaptive rolling planning procedure. It is known that in one-sector case adaptive rolling plans are efficient and good (Bala et al. 1991)). Kaganovich (1996) proved that rolling plans converge toward turnpike, [1] which is a necessary but not

* Department of Mathematical Economics, Poznań University of Economics

[1] A summary of turnpike theory can be found in McKenzie (2002).

sufficient condition for goodness. Kaganovich (1998) showed that under linear technology and maximal growth rate as optimality criterion (for constructing adaptive rolling plans) rolling plans approach von Neumann ray at (asymptotically) maximal growth rate that can be achieved among all balanced growth processes. We prove that rolling plans are good[2] under neoclassical technology of goods (theorem 5) and while proving it we use strong concavity (Vial 1983) of an indirect utility function near turnpike. To this goal, we firstly construct indirect utility function (definition). Our construction differs from a typical one (Venditti 1997) in that we express utility as a function of today's and tomorrow's inputs and not as a function of today's and tommorow's outputs (stocks of goods). Strong concavity was proven in Venditti (1997) for an economy where there is only one consumption good and all other goods are capital goods. In our case – to be in compliance with Kaganovich's approach (Kaganovich 1996) – all goods are treated as consumption/production goods at a time, so that Venditti's approach is not applicable here.[3] We also show that strong concavity of indirect utility function holds (under our assumptions) only if at most one production function is positively homogeneous of degree one and the other are subject to decreasing returns to scale.[4]

The next two sections set notation and preliminaries. In section 4 and 5 we included main results. Section 6 is a summary.

2. NOTATION AND CONVENTIONS

Symbol $\mathbf{R}^n$ denotes an n-dimensional real linear space, and $\mathbf{R}^n_+$ is its non-negative orthant. A point $x \in \mathbf{R}^n$ possesses coordinates $x_1, \ldots, x_n$. If an element of $\mathbf{R}^n$ is named x^j, where j is a nonnegative integer, then $x^j = (x_{1j}, \ldots, x_{nj})$. For $x, x' \in \mathbf{R}^n$ we write $x \geq x'$ if $x_i \geq x'_i$, $i = 1, \ldots, n$;

[2] So that, the procedure of constructing adaptive rolling plans can be used to build an evolutionary mechanism – more on this see in Bala et al. (1991) or Kaganovich (1996).

[3] We tried to prove the strong concavity of indirect utility function when its arguments were outputs – but we did not succeed because in that approach we could not determine definiteness of a counterpart of matrix $\overline{V}''$ (equation 21), which is crucial.

[4] Assumptions on production functions similar to ours were taken in Hirota and Kuga (1971), Benhabib and Nishimura (1979a), Benhabib and Nishimura (1979b), Benhabib and Nishimura (1981).

$x > x'$ means $x \geq x'$ and $x \neq x'$; $x >> x'$ is equivalent to $x_i > x'_i$, $i = 1, \ldots, n$. By $\operatorname{int} \mathbf{R}^n_+$ we denote the interior of $\mathbf{R}^n_+$, i.e. the set of positive vectors. If m and n are positive integers, then for $a \in \mathbf{R}$ symbol $a_{m \times n}$ denotes a matrix composed of m rows and n columns with a on each coordinate; a_n stands for $a_{n \times 1}$. For two matrices A, B their (right) Kronecker product is written as $A \otimes B$ (see Lancaster and Tismenetsky (1985), p. 407). The transposition of A is denoted by A^T. Euclid norm of $x \in \mathbf{R}^n$ is denoted as $\|x\|$. Writing $(x, y) \in A \times B, A \subset \mathbf{R}^n, B \subset \mathbf{R}^n$ we mean $x \in A$, $y \in B$. Given two matrices A (m rows, n columns) and B (n rows, k columns) and equation $AB = 0$, we deduce zero on right-hand-side is $0_{m \times n}$ (without writing this explicitly). Analogously: if $x \in \mathbf{R}^n$, $x \geq 0$, then zero on the right-hand-side is 0_n. The identity matrix of order n is denoted as I_n. For an element $a \in \mathbf{R}^n$ by $\operatorname{diag} a$ we denote diagonal matrix of order $n \times n$ with $a_1, \ldots, a_n$ on the diagonal. Symbol '$:=$' reads as '*by definition equals to*'.

3. PRELIMINARIES

To achieve our goal we have to give a more detailed description of a technology set Z than was done in Kaganovich (1996). Technology set Z is defined as

$$Z = \left\{ (x, y) \in \mathbf{R}^n_+ \times \mathbf{R}^n_+ : \exists x^j \in \mathbf{R}^n_+, j = 1, \ldots, n, \sum_{j=1}^n x^j \leq x, \right.$$

$$\left. \exists l_j \in \mathbf{R}_+, j = 1, \ldots, n, \sum_{j=1}^n l_j \leq 1, \; y_j \leq \sum_{j=1}^n f_j(x^j, l_j) \right\} \tag{1}$$

where $f_j : \mathbf{R}^{n+1}_+ \to \mathbf{R}_+$ is production function of j-th good, $x^j = (x_{1j}, \ldots, x_{nj})$ represents producible goods inputs and l_j stands for labour input, $j = 1, \ldots, n$. We assume for $j = 1, \ldots, n$

(i) f_j is continuous on $\mathbf{R}_+^{n+1}$, twice continuously differentiable and strictly increasing over $\operatorname{int}\mathbf{R}_+^{n+1}$ with $\dfrac{\partial f_j\,(x^j, l_j)}{\partial x_{ij}} > 0, \dfrac{\partial f_j\,(x^j, l_j)}{\partial l_j} > 0$, $i = 1, \ldots, n$, and strictly concave on interior of its domain and $f_j\,(x^j, l_j) > 0$ only if $x^j >> 0, l_j > 0$. Moreover, Hessian of f_j is negatively definite everywhere on $\operatorname{int}\mathbf{R}_+^{n+1}$.

(ii) There exists $\beta > 0$ such that if $\|x\| > \beta$, then for $(x, y) \in Z$: $y \le x$.

(iii)There exists expansible stocks vector $x \in \mathbf{R}_+^n$: $y >> x$ for some $(x, y) \in Z$.

The construction of set Z and assumption (i) guarantee that the set is closed and convex; free disposal is allowed and Z admits weak strict convexity (external effects) on inputs: $(x, y) \in Z$, $(x', y') \in Z$ with $x \ne x'$ imply that there exists $z >> \dfrac{y + y'}{2} : \left(\dfrac{x + x'}{2}, y\right) \in Z$. These properties imply that assumptions imposed on production set in Kaganovich (1996) are met, and we can use results obtained therein.

Consumption c is valuated by an instantaneous utility function $U : \mathbf{R}_+^n \to \mathbf{R}_+$ which satisfies

(iv) U is continuous, strictly concave and twice continuously differentiable on $\operatorname{int}\mathbf{R}_+^n$ with negatively defined Hessian.

(v) U is strictly increasing on $\operatorname{int}\mathbf{R}_+^n$: $c > c' \Rightarrow U(c) > U(c')$, with $\dfrac{\partial U(c)}{\partial c_j} > 0$.

Let us fix initially available input $x \in \mathbf{R}_+^n$ and a positive integer number N. A sequence $\left((x_t, y_t, c_t)\right)_{t=1}^N \subset \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ is called feasible N-process from x to $b \in \mathbf{R}_+^n$, if

$$
\begin{gathered}
(x_t, y_{t+1}) \in Z, t = 0, \ldots, N-1, \\
x_t + c_t \le y_t, t = 1, \ldots, N, \\
x_0 = x, x_N \ge b.
\end{gathered}
\tag{2}
$$

Sequence $\left((x_t, y_t, c_t)\right)_{t=1}^{\infty} \subset \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ is called feasible ∞-process from x if for all $t \geq 1$ it holds $(x_t, y_{t+1}) \in Z$, $x_t + c_t \leq y_t$ and $(x, y_1) \in Z$. An N-feasible process from $x \in \mathbf{R}_+^n$ to $b \in \mathbf{R}_+^n$ is called N-optimal from x to b if it maximizes

$$\sum_{t=1}^{N} U(c_t) \tag{3}$$

over the set of all N-feasible processes from x to b. We are interested in properties of adaptive rolling plans defined as follows. [5]

Definition 1 *Fix* $x \in \mathbf{R}_+^n$. *A sequence* $\left((x_t, y_t, c_t)\right)_{t=1}^{\infty} \subset \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ *is called adaptive rolling plan from* x *if for all* $t = 1,2,\ldots,$ *a sequence*

$$((x_t, y_t, c_t), (x_{t+1}, y_{t+1}, c_{t+1}))$$

is 2-optimal process from x_{t-1} *to* x_{t-1} *where* $x_0 = x$.

From now on we assume that there exists an adaptive rolling plan for a given initial inputs vector x_0.

Definition 2 *Triplet* $(\bar{x}, \bar{y}, \bar{c}) \in \mathbf{R}_+^n \times \mathbf{R}_+^n \times \mathbf{R}_+^n$ *is called turnpike if it is optimal solution of the following problem*

$$\max U(c)$$
$$x + c \leq y,$$
$$(x, y) \in Z,$$
$$x, y, c \in \mathbf{R}_+^n.$$

Under our assumptions turnpike exists and is unique. In what follows we denote the turnpike as $(\bar{x}, \bar{y}, \bar{c})$ and its utility as $\overline{U} = U(\bar{c})$. We also assume:

(vi) Turnpike consumption $\bar{c}$ is positive, i.e. $\bar{c} >> 0$.

We shall show that adaptive rolling plans enjoy a goodness property Gale (1967) defined as

Definition 3 *Let* $x_0 \in \mathbf{R}_+^n$. *A feasible* ∞*-process from* x_0, $\left((x_t, y_t, c_t)\right)_{t=1}^{\infty}$ *is called good if*

[5] Compare it to definitions in Bala et al. (1991) or Kaganovich (1996).

$$\liminf_{N\to\infty}\sum_{t=1}^{N}(U(c_t)-\overline{U})>-\infty. \tag{4}$$

It is known that for any ∞-process $\limsup$ of the series on left-hand-side in (4) is always finite and if lim inf is finite then the series converges (Gale 1967). Further, if a process is good then it converges to the turnpike – it is a necessary condition for goodness – and as it has been said this property holds in our setting (by results of Kaganovich 1996). Our goal is to prove that the speed of convergence toward turnpike is high enough to ensure that the condition (4) holds. We need to show that the indirect utility function (to be defined below) is twice continuously differentiable and strictly concave near the turnpike and that its Hessian is negatively definite at the turnpike.

Definition 4 *The indirect utility function* $V:\mathbf{R}_+^n\times\mathbf{R}_+^n\to\mathbf{R}_+$ *is defined as*[6]

$$\forall x,x'\in\mathbf{R}_+^n \quad V(x,x'):=\max_{\substack{y\in\mathbf{R}_+^n:\,x\le y,\\(x,y)\in Z}} U(y-x') \tag{5}$$

Certainly, function V is concave and continuous for x,x' near the turnpike $\overline{x}$.

4. STRONG CONCAVITY OF THE INDIRECT UTILITY FUNCTION NEAR TURNPIKE

Now we shall use the strength of the definition of technology set Z and assumptions. Fix $x,x'\in\mathbf{R}_+^n$. The optimization problem defining function V

$$\begin{aligned}&\max U(y-x')\\&\quad x'\le y,\\&(x,y)\in Z,\\&\quad y\in\mathbf{R}_+^n,\end{aligned} \tag{6}$$

is – by assumptions (i), (iv) – equivalent to a concave maximization problem

[6] If a set is empty then the maximum value of a function over the set equals $-\infty$, as a usual convention.

$$
\begin{aligned}
\max U(f_1\ (x^1, l_1) - x'_1, \ldots, f_1\ (x^n, l_n) - x'_n)& \\
x'_j \le f_j\ (x^j, l_j)& \\
x - \sum_{j=1}^{n} x^j \ge 0& \qquad\qquad j = 1, \ldots, n \\
\sum_{j=1}^{n} l_j \le 1& \\
x^j \in \mathbf{R}^n_+&
\end{aligned}
\tag{7}
$$

in the following sense: if $x^1, \ldots, x^n$, with some choice of $l_1, \ldots, l_n$ solves (7), then $y = (f_1\ (x^1, l_1), \ldots, f_n\ (x^n, l_n))$ solves (6) and every solution of (6) is obtained by some choice of $x^1, \ldots, x^n$ and $l_1, \ldots, l_n$ solving (7) – in fact this choice is unique (again by assumptions (i), (iv)).

4.1. Non-homogeneous case

We just keep assumption (i) in force.

Lemma 1 *There exists a neigbourhood $W \subset \mathbf{R}^n_+ \times \mathbf{R}^n_+$ of $(\bar{x}, \bar{x})$ such that function V is a twice continuously differentiable and strongly concave on W.*[7]

Proof: We divide the proof into three steps.[8]

Step 1 *Lagrange multipliers $\bar{\lambda}$ and sectoral inputs $\bar{x}^i$ are twice continuously differentiable functions of x and x'.*

We know that there is a one-to-one relationship between solutions of (6) and (7). We shall show that solution of (7) depends twice continuously differentiably on (x, x') in a neighbourhood of $(\bar{x}, \bar{x})$. Let $\bar{y}_j = f_j\ (\bar{x}^j, \bar{l}_j)$, for $\bar{x}^j, \bar{l}_j \quad j = 1, \ldots, n$, solving (7). By assumption (vi),

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[7] Symmetric matrix *A* is called negative-definite (nonpositive-definite) if all its eigenvalues are negative (nonpositive). If *A* is nonpositive-definite and is not negative-definite then we call it negative-semidefinite (Lancaster and Tismenetsky (1985), p. 179). It is known that a twice continuously differentiable concave function is strongly concave on *W* iff its Hessian is negative-definite on *W* with eigenvalues strictly separated from 0 – proof of this fact and definition of strong concavity (convexity) is contained in Vial (1983).

[8] The first one is rather standard as regards its idea, see Benhabib and Nishimura (1979a), Benhabib and Nishimura (1979b), Hirota and Kuga (1971).

$f_j(\bar{x}^j, \bar{l}_j) > \bar{x}_j$, $j = 1, \ldots, n$. Obviously, by assumptions (i) and (v), for any solution $x^1, \ldots, x^n$, $l_1, \ldots, l_n$ of (7) (under any given positive (x, x')), it holds that $x = \sum_{j=1}^{n} x^j$ and $x^j >> 0$ near $(\bar{x}, \bar{x})$, since solution of (7) depends continuously on (x, x') (by Berge's maximum theorem, Lucas and Stokey (1989), p. 62). Therefore Lagrange function for (x, x') near $(\bar{x}, \bar{x})$ can be written as

$$L(x^1, l^1, \ldots, x^n, l^n, \lambda_1, \ldots, \lambda_n, \lambda_{n+1}, x, x') =$$
$$= U(f_1(x^1, l_1) - x'_1, \ldots, f_n(x^n, l_n) - x'_n) + \sum_{i=1}^{n} \lambda_i (x_i - \sum_{j=1}^{n} x_{ij}) + \lambda_{n+1}(1 - \sum_{j=1}^{n} l_j), \tag{8}$$

where λ_i, $i = 1, \ldots, n$, denotes Lagrange multiplier. Necessary and sufficient conditions for optimality of a feasible solution $\bar{x}^1, \ldots, \bar{x}^n$, $\bar{l}_1, \ldots, \bar{l}_n$ of (7) at $(\bar{x}, \bar{x})$ read as (Takayama 1985, p. 91)

$$\begin{aligned} &\frac{\partial U(\bar{F} - \bar{x})}{\partial c_j} \frac{\partial f_j(\bar{x}^j, \bar{l}_j)}{\partial x_{ij}} - \bar{\lambda}_i = 0, \\ &\frac{\partial U(\bar{F} - \bar{x})}{\partial c_j} \frac{\partial f_j(\bar{x}^j, \bar{l}_j)}{\partial l_j} - \bar{\lambda}_{n+1} = 0, \end{aligned} \tag{9}$$

for all $i, j = 1, \ldots, n$ some positive optimal multipliers $\bar{\lambda}_i$, where $\bar{F} = F(\bar{x}^1, \bar{l}_1, \ldots, \bar{x}^n, \bar{l}_n)$ and

$$F(x^1, l_1, \ldots, x^n, l_n) = (f_1(x^1, l_1), \ldots, f_n(x^n, l_n))$$

Conditions (9) can be written in matrix notation as

$$\bar{U}'\bar{F}' - 1_{1\times n} \otimes \bar{\lambda} = 0 \tag{10}$$

where $\bar{U}'$ is the first derivative of U evaluated at $F(\bar{x}^1, \bar{l}_1, \ldots, \bar{x}^n, \bar{l}_n) - \bar{x}$, $\bar{F}' = F'(\bar{x}^1, \bar{l}_1, \ldots, \bar{x}^n, \bar{l}_n)$ and

$$F'(x^1, l_1, \ldots, x^n, l_n) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x}^1)}{\partial \mathbf{x}^1} & 0 & 0 & \ldots & 0 \\ 0 & \frac{\partial f_2(\mathbf{x}^2)}{\partial \mathbf{x}^2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{\partial f_n(\mathbf{x}^n)}{\partial \mathbf{x}^n} \end{bmatrix}$$

where

$\mathbf{x}^j = (x^j, l_j)$,

$f_j'(\mathbf{x}^j) := \frac{\partial f_j(\mathbf{x}^j)}{\partial \mathbf{x}^j} = \begin{bmatrix} \frac{\partial f_j(x^j, l_j)}{\partial x_{1j}} & \ldots & \frac{\partial f_j(x^j, l_j)}{\partial x_{nj}} & \frac{\partial f_j(x^j, l_j)}{\partial l_j} \end{bmatrix}$ and $\overline{\lambda} = [\overline{\lambda}_1 \ldots \overline{\lambda}_n \ \overline{\lambda}_{n+1}]$.

Define function $G : \mathrm{int}\mathbf{R}_+^{n(n+1)} \times \mathrm{int}\mathbf{R}_+^{n+1} \times \mathrm{int}\mathbf{R}_+^n \times \mathrm{int}\mathbf{R}_+^n \to \mathbf{R}_+^{n(n+1)} \times \mathbf{R}_+^{n+1}$ as

$$G(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^n}_{\mathbf{x}}, \underbrace{\lambda_1, \ldots, \lambda_n}_{\lambda}, x, x') = \left(U'(F(\mathbf{x}) - x')F'(\mathbf{x}) - 1_{1\times n} \otimes \lambda, x - \sum_{j=1}^{n} x^j, 1 - \sum_{j=1}^{n} l_j \right) \tag{11}$$

By assumption of optimality of $\overline{\mathbf{x}} = (\underbrace{\overline{x}^1, \overline{l}_1}_{\overline{\mathbf{x}}^1}, \ldots, \underbrace{\overline{x}^n, \overline{l}_n}_{\overline{\mathbf{x}}^n})$ at $(\overline{x}, \overline{x})$ it holds that

$$G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x}) = 0$$

and since G is of class C^1 in a neighbourhood of $(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})$, then – by the implicit function theorem – if we knew that $\frac{\partial G(\overline{\mathbf{x}}, \overline{\lambda}, \overline{x}, \overline{x})}{\partial(\mathbf{x}, \lambda)}$ were invertible then we could express $\mathbf{x}$ and λ as continuously differentiable functions of (x, x') close to $(\overline{x}, \overline{x})$. After simple transformations we get

$$\frac{\partial G(\bar{\mathbf{x}},\bar{\lambda},\bar{x},\bar{x})}{\partial(\mathbf{x},\lambda)}=\left[\begin{array}{c|c}\overline{F}'^{T}\,\overline{U}''\,\overline{F}'+((\mathrm{diag}\,\overline{U}')\otimes I_{n+1})\overline{F}'' & -1_{n\times 1}\otimes I_{n+1}\\ \hline -1_{1\times n}\otimes I_{n+1} & 0_{(n+1)\times(n+1)}\end{array}\right]=:\left[\begin{array}{c|c}A & B\\ \hline B^{T} & 0_{(n+1)\times(n+1)}\end{array}\right]$$

(12),

where $\overline{U}''=U''(F(\bar{\mathbf{x}})-\bar{x})$ is Hessian of U evaluated at $F(\bar{\mathbf{x}})-\bar{x}$, $\overline{F}''=F''(\bar{\mathbf{x}})$ and

$$F''(\bar{\mathbf{x}})=\mathrm{diag}\left(f_1''(\bar{\mathbf{x}}^1),\ldots,f_n''(\bar{\mathbf{x}}^n)\right)$$

is a block-diagonal matrix of rank $(n+1)^2$ with Hessians $f_1''(\bar{\mathbf{x}}^1),\ldots,f_n''(\bar{\mathbf{x}}^n)$ on the diagonal. We shall show first that A (defined in (12)) is a negative definite matrix. Certainly $\overline{U}''$ is a negative definite and so is $\overline{F}''$. Further $((\mathrm{diag}\,\overline{U}')\otimes I_{n+1})\overline{F}''$ is negative definite by assumption (i), so that A is negative definite. It is easily seen that for any $0\neq\mathbf{x}\in\mathbf{R}^{n(n+1)}$ which satisfies $(-1_{1\times n}\otimes I_{n+1})\mathbf{x}=0$, it holds that $\mathbf{x}^T\dfrac{\partial^2 L(\bar{\mathbf{x}},\bar{\lambda},\bar{x},\bar{x})}{\partial\mathbf{x}^2}\mathbf{x}<0$.

Therefore matrix (12) is non-singular. Further,

$$\frac{\partial G(\bar{\mathbf{x}},\bar{\lambda},\bar{x},\bar{x})}{\partial(x,x')}=\begin{bmatrix}0_{n(n+1)\times n} & -\overline{F}'^{T}\,\overline{U}''\\ I_n & 0_{n\times n}\\ 0_{1\times n} & 0_{1\times n}\end{bmatrix}\tag{13}$$

By the implicit function theorem (Nikaido 1968, p. 85) there exists a neighbourhood W of $(\bar{x},\bar{x})$ and continuously differentiable function $g:W\to\mathrm{int}\,\mathbf{R}_+^{n(n+1)}\times\mathrm{int}\,\mathbf{R}_+^n$ such that

$\forall(x,x')\in W:\quad G(g(x,x'),x,x')=0$ and if $g(x,x')\neq(\mathbf{x},\lambda)\in W$ then $G(\mathbf{x},\lambda,x,x')\neq 0$ and it follows that $\mathbf{x}$ does not solve (7) at (x,x'). Since $g(x,x')\in\mathrm{int}\,\mathbf{R}_+^{n(n+1)}\times\mathrm{int}\,\mathbf{R}_+^n$ it holds $\forall(x,x')\in W$:

$$V(x,x')=U(F(\bar{\mathbf{x}}(x,x'))-x'),\tag{14}$$

$$\frac{\partial g(\bar{x},\bar{x})}{\partial(x,x')}=-\left[\frac{\partial G(\bar{\mathbf{x}},\bar{\lambda},\bar{x},\bar{x})}{\partial(\mathbf{x},\lambda)}\right]^{-1}\frac{\partial G(\bar{\mathbf{x}},\bar{\lambda},\bar{x},\bar{x})}{\partial(x,x')},\tag{15}$$

where $g(x,x')=(\bar{\mathbf{x}}(x,x'),\bar{\lambda}(x,x'))$.

By the envelope theorem (see Takayama 1985, p. 138) we get from (8) $\forall (x,x')\in W$

$$\frac{\partial V(x,x')}{\partial(x,x')}=\left(\overline{\lambda}_1\,(x,x'),\dots,\overline{\lambda}_n\,(x,x'),-U'(F(\overline{\mathbf{x}}(x,x'))-x')\right) \tag{12}$$

and since $\overline{\lambda}(\cdot,\cdot),\overline{\mathbf{x}}(\cdot,\cdot)$ are continuously differentiable on W then V is twice continuously differentiable in W.

Step 2 *Hessian of the indirect utility function*

All we need now is to show that Hessian $V''(\overline{x},\overline{x})$ is negative definite. It holds (arguments omitted in the last line)

$$\overline{V}'':=V''(\overline{x},\overline{x})=\begin{bmatrix}\dfrac{\partial(\overline{\lambda}_1,\dots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x} & \dfrac{\partial(\overline{\lambda}_1,\dots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x'}\\ -\dfrac{\partial U'(F(\overline{\mathbf{x}}(\overline{x},\overline{x}))-x')}{\partial x} & -\dfrac{\partial U'(F(\overline{\mathbf{x}}(\overline{x},\overline{x}))-x')}{\partial x'}\end{bmatrix}=$$

$$=\begin{bmatrix}\dfrac{\partial(\overline{\lambda}_1,\dots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x} & \dfrac{\partial(\overline{\lambda}_1,\dots,\overline{\lambda}_n)(\overline{x},\overline{x})}{\partial x'}\\ -\overline{U}''\overline{F}'\dfrac{\partial\overline{\mathbf{x}}(\overline{x},\overline{x})}{\partial x} & -\overline{U}''\overline{F}'\dfrac{\partial\overline{\mathbf{x}}(\overline{x},\overline{x})}{\partial x'}+\overline{U}''\end{bmatrix}=$$

$$=\begin{bmatrix}0_{n\times n(n+1)} & I_n & 0_{n\times 1}\\ -\overline{U}''\overline{F}' & 0_{n\times n} & 0_{n\times 1}\end{bmatrix}\begin{bmatrix}\dfrac{\partial\overline{\mathbf{x}}}{\partial x} & \dfrac{\partial\overline{\mathbf{x}}}{\partial x'}\\ \dfrac{\partial\overline{\lambda}}{\partial x} & \dfrac{\partial\overline{\lambda}}{\partial x'}\end{bmatrix}+\begin{bmatrix}0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & \overline{U}''\end{bmatrix} \tag{17}$$

By (14), (15) and (17)

$$\overline{V}''=\begin{bmatrix}0_{n\times n(n+1)} & I_n & 0_{n\times 1}\\ -\overline{U}''\overline{F}' & 0_{n\times n} & 0_{n\times 1}\end{bmatrix}\frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(\mathbf{x},\lambda)}\begin{bmatrix}0_{n(n+1)\times(n+1)} & -\overline{F}'^T\,\overline{U}''\\ I_n & 0_{n\times n}\\ 0_{1\times n} & 0_{1\times n}\end{bmatrix}+\begin{bmatrix}0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & \overline{U}''\end{bmatrix} \tag{18}$$

By (12) we can write

$$\left[\frac{\partial G(\overline{\mathbf{x}},\overline{\lambda},\overline{x},\overline{x})}{\partial(\mathbf{x},\lambda)}\right]^{-1}=\begin{bmatrix}C & D\\ D^T & E\end{bmatrix} \tag{19}$$

where[9]

$$
\begin{aligned}
E &:= -(B^T A^{-1} B)^{-1},\\
D &:= -A^{-1} BE,\\
C &:= A^{-1} + A^{-1} BEB^T A^{-1},
\end{aligned}
\tag{20}
$$

and by (18) we get

$$
\overline{V}'' = \left[\begin{array}{c|c} \dfrac{-\begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} E \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix}}{\overline{U}''\overline{F}' D \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix}} & \dfrac{\begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} D^T \overline{F}'^T \overline{U}''}{-\overline{U}''\overline{F}' C \overline{F}'^T \overline{U}''} \end{array}\right] + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \overline{U}'' \end{bmatrix} \tag{21}
$$

Step 3 *Negative definiteness of Hessian*

Since V is concave, then $\overline{V}''$ is at least nonpositive definite. To show that $\overline{V}''$ is negative definite we need to prove that it is non-singular. Suppose that there exists $(x, x') \in \mathbf{R}^n \times \mathbf{R}^n$ such that

$$
-\begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} E \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} x + \begin{bmatrix} I_n & 0_{n\times 1}\end{bmatrix} D^T \overline{F}'^T \overline{U}'' x' = 0,
$$

$$
\overline{U}''\overline{F}' D \begin{bmatrix} I_n \\ 0_{1\times n}\end{bmatrix} x - \overline{U}''\overline{F}' C \overline{F}'^T \overline{U}'' x' + \overline{U}'' x' = 0.
$$

This system of equations is equivalent (by (20)) to

$$
-E\begin{bmatrix} x \\ 0\end{bmatrix} - EB^T A^{-1} \overline{F}'^T \overline{U}'' x' = \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}, \tag{22}
$$

$$
-\overline{U}''\overline{F}' A^{-1} BE \begin{bmatrix} x \\ 0\end{bmatrix} - \overline{U}''\overline{F}' A^{-1} \overline{F}'^T \overline{U}'' x' - \overline{U}''\overline{F}' A^{-1} BEB^T A^{-1} \overline{F}'^T \overline{U}'' x' + \overline{U}'' x' = 0, \tag{23}
$$

---

[9]See (12)

where a is some real number. Substituting $E\begin{bmatrix} x \\ 0 \end{bmatrix}$ into (23) we get

$$\overline{U}''\overline{F}'A^{-1}B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix} - \overline{U}''\overline{F}'A^{-1}\overline{F}'^{T}\,\overline{U}''x' + \overline{U}''x' = 0,$$

which is equivalent to (after left-multiplying by $\overline{F}'^{T}$)[10]

$$RA^{-1}\left(\overline{F}'^{T}\,\overline{U}''x' - B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}\right) = -B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}, \tag{24}$$

where

$$R := (\operatorname{diag}(\overline{U}') \otimes I)\overline{F}''. \tag{25}$$

By invertibility of R, A and since $AR^{-1} = I + \overline{F}'^{T}\,\overline{U}''\overline{F}'R^{-1}$ we get from (24)

$$\overline{F}'^{T}\,\overline{U}''x' = -\overline{F}'^{T}\,\overline{U}''\overline{F}'R^{-1}B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix},$$

which is equivalent to

$$x' = -\overline{F}'R^{-1}B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}.$$

Putting x' into (22), observing that $EB^{T}A^{-1}\overline{F}'^{T}\,\overline{U}''\overline{F}'R^{-1}B = EB^{T}R^{-1}B - I_{n}$ and due to invertibility of E we get

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = B^{T}R^{-1}B\begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix}, \tag{26}$$

But by definition of R (see (25)) R^{-1} is a block-diagonal matrix with negative definite matrices $\left[\dfrac{\partial U(F(\mathbf{x}) - \overline{x})}{\partial c_{j}} f_{j}''(\overline{x}^{j})\right]^{-1}$ on the diagonal.

Moreover $B = -1_{n\times 1} \otimes I_{n+1}$ and (26) imply

---

[10] Since partial derivatives of f_i at $\overline{x}^{i}$ are positive and $\overline{F}'^{T}\,\overline{U}''\overline{F}' = A - R$.

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \left(\sum_{j=1}^{n} \left[\frac{\partial U(F(\bar{\mathbf{x}}) - \bar{x})}{\partial c_j} f_j''(\bar{x}^j) \right] \right)^{-1} \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix},$$

which is possible only if $a = 0, x = 0$. From this we get $x' = 0$ (by the equation above to (26)), so that we have shown that equality $\overline{V}''\begin{bmatrix} x \\ x' \end{bmatrix} = 0$ (see (21)) is possible only if $x = x' = 0$, therefore $\overline{V}''$ is negative definite and w.l.o.g. we can assume that $V''(x, x')$ is negative definite on W, which ends the proof.

Remark 1 *It should be noted that the above proof „works" for all points* $(x, x') \in \operatorname{int}\mathbf{R}_+^n \times \operatorname{int}\mathbf{R}_+^n$ *for which optimal consumption (see (7)) is positive. This observation allows us to broaden the class of 'base' models for which the indirect utility function is strongly concave: models of type (7) generate strongly concave indirect utility functions if assumptions (i)-(vi) are met and for every* $(x, x') \in \operatorname{int}\mathbf{R}_+^n \times \operatorname{int}\mathbf{R}_+^n$ *it holds that optimal consumption level is positive.*[11]

4.2. Homogeneous production functions

Let us put aside the assumption of strict concavity and negative definiteness of Hessians of production functions. Suppose that for at least two *j*'s (w.l.o.g. $j = 1, 2, \ldots$) it holds

(vii) f_j is continuous on $\mathbf{R}_+^{n+1}$, positively homogeneous of degree 1 and twice continuously differentiable and strictly increasing over $\operatorname{int}\mathbf{R}_+^{n+1}$ with $\dfrac{\partial f_j(x^j, l_j)}{\partial x_{ij}} > 0, \dfrac{\partial f_j(x^j, l_j)}{\partial l_j} > 0$, $i = 1, \ldots, n$ and concave on interior of its domain and $f_j(x^j, l_j) > 0$ only if $x^j \gg 0, l_j > 0$. Moreover, rank of negatively semidefinite Hessian of f_j is n everywhere on $\operatorname{int}\mathbf{R}_+^{n+1}$.

---

[11] Our approach eliminates inputs x, x' and consumption $c = y - x'$ with 0 entries – corner solutions are excluded.

We shall proceed keeping in mind that – by Euler's theorem (Lancaster 1968, p. 335-336) – if f_j satisfies assumption (4.2) then

$$f_j(\mathbf{x}^j) = f_j^{'}(\mathbf{x}^j)\mathbf{x}^j$$

$$y^T f_j^{"}(\mathbf{x}^j) y = 0 \Leftrightarrow \exists \lambda \in \mathrm{R} : y = \lambda \mathbf{x}^j$$

where $\mathbf{x}^j = (x^j, l_j) \in \operatorname{int} \mathbf{R}_+^n \times \operatorname{int} \mathbf{R}_+$.

We shall show that if functions f_1 and f_2 fulfill (4.2) (and the other ones satisfy (i) or (4.2)) then Hessian of the indirect utility function V is nonpositive definite. Since the way of construction of V is as before, then to show that V is twice continuously differentiable (near $(\bar{x}, \bar{x})$) it is sufficient to show that matrix A (see (12)) is negative definite. Certainly, A is nonpositive definite. It is negative semidefinite if singular. Suppose that for some $0 \neq \mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \in \mathbf{R}_+^{n(n+1)}$: $A\mathbf{x} = 0$. It implies $\mathbf{x}^T A \mathbf{x} = 0$, which is possible only if $\mathbf{x}^T \overline{F}'^T \overline{U}'' \overline{F}' \mathbf{x} = 0$ and $\mathbf{x}^T \overline{F}'^T = 0$. Since $f_j^{"}(\overline{\mathbf{x}}^j)$, $j = 3, \ldots, n$, are negatively defined and by construction of $\overline{F}''$, then $\mathbf{x}^3 = \ldots = \mathbf{x}^n = 0$. By Euler's theorem and assumption (4.2) on rank of Hessian $f_j^{"}$ there exist scalars α_j such that $\mathbf{x}^j = \alpha_j \overline{\mathbf{x}}^j$, j=1,2 (this observation comes from Hirota and Kuga (1971)). Since $\mathbf{x}^T \overline{F}'^T \overline{U}'' \overline{F}' \mathbf{x} = 0$ only if $\overline{F}' \mathbf{x} = 0$, then using Euler's theorem again brings

$$0 = \frac{\partial f_j(\overline{\mathbf{x}}^j)}{\partial \mathbf{x}} \overline{\mathbf{x}}^j = \alpha_j f_j(\overline{\mathbf{x}}^j),$$

which is possible only if $\alpha_j = 0$, since $f_j(\overline{\mathbf{x}}^j) > 0$ by assumption. This implies $\mathbf{x} = 0$ – contradiction, so that A is non-singular, and therefore negatively definite. We can use (21) to express $\overline{V}''$. Hessian $\overline{V}''$ is negatively definite if solution *x*,*x*' of (22), (23) (or equivalently (22), (24)) is trivial (if it exists for a given value of *a*). We know that turnpike labour inputs are positive: $\bar{l}^j > 0$, $j = 1, \ldots, n$. Take any α_1, α_2 non-vanishing simultaneously, such that $\alpha_1 \bar{l}_1 + \alpha_2 \bar{l}_2 = 0$. Let us define $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \in \mathbf{R}_+^{n(n+1)}$ as $\mathbf{x}^j = \alpha_j \overline{\mathbf{x}}^j, j = 1, 2,$ $\mathbf{x}^j = 0, j = 3, \ldots, n$. Put $a = 0$, $x' = (\alpha_1 f_1(\mathbf{x}^1), \alpha_2 f_2(\mathbf{x}^2), 0, \ldots, 0) \in \mathbf{R}_+^n$, and

$x \in \mathbf{R}^n$ $\begin{bmatrix} x \\ 0 \end{bmatrix} = \alpha_1 \bar{\mathbf{x}}^1 + \alpha_2 \bar{\mathbf{x}}^2$. Substituting the values into system (22), (24) and observing that $x' = [\alpha_1 \quad \alpha_2 \quad 0 \quad \ldots \quad 0]\bar{F}'\mathbf{x}$, $\bar{F}'^T \bar{U}'' \bar{F}' = A - R$ and E is non-singular we see that x, x' solve the system for $\alpha = 0$ and $x' \neq 0$. This means that $\bar{V}''$ is singular and therefore negative semidefinite.

Suppose now that only f_1 satisfies (4.2) and $f_2, \ldots, f_n$ satisfy (i). We shall show first that system (22), (24) has solution if $a = 0$. Let $\mathbf{x} = (\bar{\mathbf{x}}^1, 0, \ldots, 0) \in \mathbf{R}_+^{n(n+1)}$. Left-multiplying (24) by $\mathbf{x}^T$ we get

$$0 = \mathbf{x}^T R A^{-1} \left(\bar{F}'^T \bar{U}'' x' - B \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix} \right) = -\mathbf{x}^T B \begin{bmatrix} 0_{n\times 1} \\ a \end{bmatrix} = a\bar{l}_1 ,$$

which is possible only if $a = 0$. So that (22), (24) become

$$\begin{aligned} &-\begin{bmatrix} x \\ 0 \end{bmatrix} - BA^{-1}\bar{F}'^T \bar{U}'' x' = 0, \\ &RA^{-1}\bar{F}'^T \bar{U}'' x' = 0, \end{aligned} \tag{27}$$

which – by Euler's theorem – implies $A^{-1}\bar{F}'^T \bar{U}'' x' = (\alpha\bar{\mathbf{x}}^1, 0_n, \ldots, 0_n)$ for some $\alpha \in \mathbf{R}$ and

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = -B^T A^{-1} \bar{F}'^T \bar{U}'' x' = -\alpha\bar{\mathbf{x}}^1 .$$

Since $\bar{l}_1 > 0$ it can hold only if $x = 0$, $\alpha = 0$, but this shows that system (22), (24) has only a trivial solution, under $\alpha = 0$ (if $a \neq 0$ then it is an inconsistent system). We have shown that $\bar{V}'' \begin{bmatrix} x \\ x' \end{bmatrix} = 0$ if $x = x' = 0$, so that Hessian $\bar{V}''$ is negatively definite. Now we can state[12]

Theorem 1 *Suppose assumptions (ii)-(vi) hold and production function f_j satisfies assumption (i) or (4.2), $j = 1, \ldots, n$. The indirect utility function*

---

[12] Similar results, but for social production frontier only (not for utility), were derived in Lancaster (1968), p. 127-133.

V (see (6)) is strongly concave in a neighbourhood of turnpike if and only if the number of production functions satisfying (4.2) is less than 2.

5. ROLLING PLANS ARE GOOD

From Kaganovich (1996) we know that for every rolling plan $\left((x_t, y_t, c_t)\right)_{t=1}^{\infty}$ it holds[13]

$$\lim_{t\to\infty}(x_t, y_t, c_t) = (\bar{x}, \bar{y}, \bar{c}) \tag{28}$$

To prove that a rolling-plan is good we shall show that it converges toward turnpike fast in a neighbourhood of turnpike. The main result of the paper is

Theorem 2 *Fix $x \in \mathbf{R}_+^n$. Suppose that $\left((x_t, y_t, c_t)\right)_{t=1}^{\infty}$ is an adaptive rolling plan from x. The sequence is a good process.*

Proof: By (28) we have $\lim_{t\to\infty} x_t = \bar{x}$. There exists a neighbourhood W' of $\bar{x}$ such that $W' \times W' \subset W$ *where* W satisfies the thesis of lemma 4.1. Since $\bar{x}$ is strictly positive, then for sufficiently large t's x_{t+1} solves uniquely

$$\max_{x' \in W'} \left\{ V(x_t, x') + V(x', x_t) \right\} \tag{29}$$

To prove the thesis it suffices (by concavity of V) to show that mapping $x \mapsto \arg\max\left\{V(x,x') + V(x',x) : x \in W'\right\}$ is contractive at $\bar{x}$.[14] Since restriction $V\big|_W$ is a function of C^2-class over W it must hold for large t: $x_{t+1} \in W'$ and $\frac{\partial V(x_t, x_{t+1})}{\partial x'} + \frac{\partial V(x_{t+1}, x_t)}{\partial x} = 0$. Let us define a function $S : W' \times W' \to \mathbf{R}^n$ S as follows:

$$\forall (x, x') \in W' \times W' \qquad S(x, x') = \frac{\partial V(x, x')}{\partial x'} + \frac{\partial V(x', x)}{\partial x}$$

---

[13] After some mild modification of proof of theorem 1, p. 181, in Kaganovich (1996).

[14] For a neighbourhood W of $\bar{x}$ we call mapping $h : W' \to W'$ contractive at $\bar{x}$ if $\exists \alpha \in [0,1) \forall x \in W : \quad \left\| h^q(x) - h^q(\bar{x}) \right\| \le \alpha \left\| x - \bar{x} \right\|$, where q is a fixed positive integer number and $h^q(x) := \underbrace{h \circ \ldots \circ h}_{q\ times}(x)$.

Then

$$\frac{\partial S(x,x')}{\partial x} = \frac{\partial^2 V(x,x')}{\partial x \partial x'} + \frac{\partial^2 V(x',x)}{\partial x \partial x'}^T$$

$$\frac{\partial S(x,x')}{\partial x'} = \frac{\partial^2 V(x,x')}{\partial x'^2} + \frac{\partial^2 V(x',x)}{\partial x^2}$$

For every $(x,x') \in W' \times W''$ it holds that $\frac{\partial S(x,x')}{\partial x'}$ is an invertible matrix and therefore, since $S(\bar{x},\bar{x}) = 0$, then there exists[15] a function $h : W' \to W'$ such that $\forall x \in W : S(x,h(x)) = 0$ and h is continuously differentiable on W'. Moreover

$$h'(\bar{x}) = -\left[\frac{\partial S(\bar{x},\bar{x})}{\partial x'}\right]^{-1} \frac{\partial S(\bar{x},\bar{x})}{\partial x}.$$

Denote $V_{21} = \frac{\partial^2 V(\bar{x},\bar{x})}{\partial x \partial x'}, V_{11} = \frac{\partial^2 V(\bar{x},\bar{x})}{\partial x^2}, V_{22} = \frac{\partial^2 V(\bar{x},\bar{x})}{\partial x'^2}$. We shall show that $-[V_{11}+V_{22}]^{-1}[V_{21}^T + V_{21}]$ possesses no eigenvalue with modulus greater or equal to one – this will finish the proof, since then h is contractive at $\bar{x}$. By symmetry of $V_{11}+V_{22}$ and $V_{21}^T + V_{21}$ the eigenvalues of interest are real. Suppose that for some $0 \neq x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ it holds

$$-[V_{11}+V_{22}]^{-1}[V_{21}^T + V_{21}]x = \lambda x,$$

which is equivalent to

$$-[V_{21}^T + V_{21}]x = \lambda [V_{11}+V_{22}]x.$$

Left-multiplying last equality by x^T, and using negative definiteness of $V_{11}+V_{22}$ we get

$$\lambda = -\frac{x^T[V_{21}^T + V_{21}]x}{x^T[V_{11}+V_{22}]x} \tag{30}$$

By assumption, $\bar{V}'' = V''(\bar{x},\bar{x}) = \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix}$ is negatively definite so that we have

[15] If necessary, we can choose an open subset $W'' \subset W'$ with $\bar{x} \in W''$ instead of W'.

$$0 > \begin{bmatrix} x^T & -x^T \end{bmatrix} \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} x \\ -x \end{bmatrix} = x^T[V_{11} + V_{22}]x - x^T[V_{21}^T + V_{21}]x.$$

and

$$0 > \begin{bmatrix} x^T & x^T \end{bmatrix} \begin{bmatrix} V_{11} & V_{21}^T \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = x^T[V_{11} + V_{22}]x + x^T[V_{21}^T + V_{21}]x.$$

Therefore

$$x^T[V_{11} + V_{22}]x < x^T[V_{21}^T + V_{21}]x < -x^T[V_{11} + V_{22}]x$$

and

$$-1 < \frac{x^T[V_{21}^T + V_{21}]x}{x^T[V_{11} + V_{22}]x} < 1,$$

which shows that $|\lambda| < 1$ (see 30). This ends the proof.

6. SUMMARY

In this paper we have shown that adaptive rolling plans are good under the assumption of neoclassical technology. We have also shown (by use of rather elementary tools) strong concavity of indirect utility function. As we mentioned, in Bala et al. (1991) it was proven that in one-sector case adaptive rolling plans are good and efficient. ”Efficiency puzzle” of adaptive rolling plans in multiproduct economy seems to have been unsolved, so far.

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